

NETWORK REDUCTION PROBLEMS

Anna ČERNÁ¹, Jan ČERNÝ², Štefan PEŠKO³, Peter CZIMMERMANN⁴

^{1, 2} University of Economics, Faculty of Management, Jindřichův Hradec, Czech Republic

^{3, 4} University of Žilina, Faculty of Management and Informatics, Žilina, Slovakia

e-mail: cerny@fm.vse.cz, cerna@fm.vse.cz,
pesko@fria.fri.uniza.sk, czimmerman@fria.fri.uniza.sk

Abstract

In this paper we deal with some problems that occur in reduction of transport networks. Each of the studied problems can be represented as a finding of minimal subgraph with given properties. We also deal with the complexity of mentioned problems.

Keywords: Network, subnetwork, optimization, complexity.

1 INTRODUCTION

In practice, one can often meet a decision problem how to choose an optimal sub-network of the given network. Let us present several examples.

P1. Subnetwork of Rural Road Network

In Czech and Slovak Republics, it happens that, because of some historical reasons, the road network connecting small villages in some region is "too dense". It may mean that e.g.

- a) it is too expensive to reconstruct all the roads to keep some new standards,
- b) it is impossible to maintain all the roads carriageable in the winter time.

Of course, all the villages ought to remain accessible from the others. Usually, there exists some limit of the total network length. The problem is to find an optimal subnetwork within the limit. The objective function expresses lengthening of the distances on the reduced network

P2. Subnetwork of Urban Street Network

Suppose that there exists a network of streets suitable for bus transport in a small or medium town. However, if the bus service of the town operates on the whole network

then the service will become too extensive - many routes will have big headways between following buses at the stops. Then the municipal administration and the transport company management are looking for some reduction of network which can increase the frequency of buses from remaining stops. Of course, the sources and sinks of strong passenger flows ought to remain untouched and their routes must not be much longer than on the original unreduced network.

P3. Trolleybus Subnetwork of a Bus Network

In medium towns, the usual development of public transport starts by the phase of bus transport and then, after many years, the idea of partial electrification of streets for trolleybus service arises. The problem is how to choose a trolleybus subnetwork of the bus network. Naturally, the trolleybuses have to serve to the strongest passenger flows.

Even though these three examples are connected with road traffic one can say that the other types of transport may encounter similar "network reduction" problems as well.

P4. Reduction of Railway Network

In both Czech and Slovak Republics, public administration together with railway authorities have been trying to answer the question whether to abandon the public railway service on some of the regional tracks.

P5. Reduction of Waterway Network

It may happen that a waterway network in some area needs some modernization which is too expensive if applied totally. The authorities have to choose a suitable subnetwork for it.

The authors imagine that similar problems can arise e.g. in some forest-road network or mountain ski tow and cable way network modernization, etc.

One cannot say that no subnetwork optimization has been treated yet in the mathematical letters. We can mention e.g. the minimum spanning tree problem. However, it is hardly applicable to the above mentioned problems.

2 OPTIMIZATION PROBLEMS

All the above mentioned examples lead to a family of **mathematical optimization problems**. Their common feature is that there exists a connected non-oriented finite graph $G = (V, E, d)$ with the length $d(e)$ for each edge $e \in E$. This graph represents a network. As usual, the value

$$d(p) = d(v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n) = d(e_1) + \dots + d(e_n)$$

represents the length of the path $p = (v_0, e_1, v_1, e_2, v_2, \dots, v_{n-1}, e_n, v_n)$ and, on the graph G , the distance between the vertices v and w is denoted by $d(v, w)$ (i.e. the length

of the shortest path from v to w). Of course, the value $d(e)$ need not represent the distance in kilometres or miles. It can express the passing time or financial expenses as well.

OP1. Admissible Lengthening of Routes

Given the graph G and a number $q \in \langle 1; \infty \rangle$, usually $q \approx 1.1 - 1.5$. To find a connected spanning subgraph $G_q = (V, E_q, d_q)$ of G such that

$$d_q(e) = d(e) \text{ for each } e \in E_q,$$

$$d_q(v, w) \leq q \cdot d(v, w) \text{ for each pair } (v, w) \in V \times V,$$

$$d(E_q) = \sum_{e \in E_q} d(e) \rightarrow \min.$$

OP2. Admissible Lengthening of Important Routes

Given the graph G , the set of "important pairs" $W \times W, W \subset V$ and a number $q \in \langle 1; \infty \rangle$, usually $q \approx 1.1 - 1.5$. To find a subgraph $G_{qW} = (V, E_{qW}, d_{qW})$ of G such that

$$d_{qW}(e) = d(e) \text{ for each } e \in E_{qW},$$

$$d_{qW}(v, w) \leq q \cdot d(v, w) \text{ for each pair } (v, w) \in W \times W,$$

$$d(E_{qW}) = \sum_{e \in E_{qW}} d(e) \rightarrow \min.$$

Remark 1 : The second constraint implies that for each pair $(v, w) \in W \times W$ the vertex w is accessible from the vertex v on the graph G_{qW} since otherwise $d_{qW}(v, w) = \infty$.

OP3. Admissible Lengthening of Important Routes – A General Version

Given the graph G , the set of "important pairs" $D \subset V \times V$ and a number $q \in \langle 1; \infty \rangle$, usually $q \approx 1.1 - 1.5$. To find a subgraph $G_{qD} = (V, E_{qD}, d_{qD})$ of G such that

$$d_{qD}(e) = d(e) \text{ for each } e \in E_{qD},$$

$$d_{qD}(v, w) \leq q \cdot d(v, w) \text{ for each pair } (v, w) \in D,$$

$$d(E_{qD}) = \sum_{e \in E_{qD}} d(e) \rightarrow \min.$$

OP4. Limited Total Length of Complete Subnetwork

Given the graph G and let d_S be the length of the minimum spanning tree of G . Let $\delta \in \langle d_S; d(E) \rangle$ represents the limit of the subnetwork length where $d(E) = \sum_{e \in E} d(e)$. To find a connected spanning subgraph $G_\delta = (V, E_\delta, d_\delta)$ of G such that

$$\begin{aligned}
d_\delta(e) &= d(e) \text{ for each } e \in E_\delta, \\
d(E_\delta) &= \sum_{e \in E_\delta} d(e) \leq \delta, \\
\max \left\{ \frac{d_\delta(v, w)}{d(v, w)} \mid (v, w) \in V \times V \right\} &\rightarrow \min.
\end{aligned}$$

OP5. Limited Total Length of Complete Subnetwork – 2nd Version

Given the graph G and let d_S be the length of the minimum spanning tree of G . Let $\delta \in \langle d_S; d(E) \rangle$ represents the upper bound of the subnetwork length where $d(E) = \sum_{e \in E} d(e)$. Let $f(v, w) \geq 0$ represents the flow from the vertex v to the vertex w , $(v, w) \in V \times V$. To find a connected spanning subgraph $G_\delta = (V, E_\delta, d_\delta)$ of G such that

$$\begin{aligned}
d_\delta(e) &= d(e) \text{ for each } e \in E_\delta, \\
d(E_\delta) &= \sum_{e \in E_\delta} d(e) \leq \delta, \\
\sum_{(v, w) \in V \times V} f(v, w) d_\delta(v, w) &\rightarrow \min.
\end{aligned}$$

OP6. Limited Total Length of Partial Subnetwork

Given the graph G and a subset $W \subset V$. Let d_S be the length of the minimum spanning tree of G . Let $\delta \in \langle d_S; d(E) \rangle$ where $d(E) = \sum_{e \in E} d(e)$. To find a subgraph $G_{\delta W} = (V, E_{\delta W}, d_{\delta W})$ of G such that

$$\begin{aligned}
d_{\delta W}(e) &= d(e) \text{ for each } e \in E_{\delta W}, \\
d(E_{\delta W}) &= \sum_{e \in E_{\delta W}} d(e) \leq \delta, \\
\max \left\{ \frac{d_{\delta W}(v, w)}{d(v, w)} \mid (v, w) \in W \times W \right\} &\rightarrow \min.
\end{aligned}$$

Remark 2 : If the reached minimum in the condition of optimality is finite then for each pair $(v, w) \in W \times W$ the vertex w is accessible from the vertex v on the graph $G_{\delta W}$ since otherwise $d_{\delta W}(v, w) = \infty$. If the minimum is ∞ , then there exists an unconnected pair $(v, w) \in W \times W$ and such a value δ is infeasible.

Remark 3 : It is not necessary to study the 2nd variant of this problem, it is covered by the problem OP4 if $f(v, w) = 0$ for all pairs $(v, w) \notin W \times W$.

OP7. Limited Total Length of Complete Subnetwork - 3rd Version

Given the graph G , let $t(e)$ be another attribute of the edge e (e.g. d may represent the running time and t the walking time) and $t(e) \geq d(e)$ for each $e \in E$. Let d_S be the length of the minimum spanning tree of G . Let $\delta \in \langle d_S; d(E) \rangle$ represents the upper bound of the subnetwork length where $d(E) = \sum_{e \in E} d(e)$. Let $f(v, w) \geq 0$ represents the flow from the vertex v to the vertex w , $(v, w) \in V \times V$. To find a connected spanning subgraph $G_\delta = (V, E_\delta, d_\delta)$ of G such that if we define

$$d_\delta(e) = d(e) \text{ for each } e \in E_\delta \text{ and } d_\delta(e) = t(e) \text{ for each } e \in E - E_\delta,$$

$$d_\delta(v, w) \text{ is length of the shortest path from } v \text{ to } w \text{ on the graph } G_{\delta E} = (V, E, d_\delta)$$

then we have

$$d(E_\delta) = \sum_{e \in E_\delta} d(e) \leq \delta,$$

$$\sum_{(v,w) \in V \times V} f(v, w) d_\delta(v, w) \rightarrow \min,.$$

3 SOLVABILITY AND COMPLEXITY OF PROBLEMS

Any of the problems OP1-OP7 has the following general structure: There is given a graph $G = (V, E)$ with some attribute or attributes of the edges from E . The goal is to find a subset $E' \subset E$ fulfilling some constraints and minimizing an objective function. It is obvious that each of these problems is solvable by a suitable algorithm of the "back-tracking" or "branch-and-bound" type, starting with the set $E' = E$ and subsequent omitting individual edges from this set keeping the constraints valid and improving the objective function. However, this approach is feasible for small size problems. Whether there may exist a more fast exact optimal algorithm depends on the complexity of these problems.

OP1. Admissible Lengthening of Routes

It is possible to show that this problem is NP -hard for $q \geq 1$ (see for example [2]).

OP3. Admissible Lengthening of Important Routes – A General Version (ALIR-GV)

We will show that this problem is NP -hard for $q = 1$ (a proof for $q > 1$ is more complicated and it will be published soon). In our proof we will use the problem "Hitting set":

Definition. ([3] p.64) Let a collection $S = \{S_1, \dots, S_p\}$ of subsets of a set U be given. The problem to find a minimal subset U' of the set U such that $S_i \cap U' \neq \emptyset$

for $\forall i \in \{1, \dots, p\}$ is called the hitting set problem.

R.M.Karp proved in [4] that the hitting set problem is *NP*-hard.

Theorem. The problem (*ALIR* – *GV*) is *NP*-hard for $q = 1$.

Proof. We will show that there is a polynomial reduction from the hitting set problem. Let the set $U = \{v_1, v_2, \dots, v_n\}$ and the collection $S = \{S_1, S_2, \dots, S_p\}$ of its non-empty subsets be given. We construct the graph $G = (V, E)$, where

$$V = \{S_1, \dots, S_p, S'_1, \dots, S'_p, v_1, \dots, v_n, v'_1, \dots, v'_n\},$$

$$E = \{\{S_i, v_j\}, \{S'_i, v'_j\} \mid \text{if } v_j \in S_i\} \cup \{\{v_j, v'_j\} \mid \text{for } j = 1, 2, \dots, n\}.$$

Let $d(e) = 1$ for each $e \in E$ and $D = \{(S_i, S'_i) \mid \text{for } i = 1, 2, \dots, p\}$. Let $G_{1D} = (V, E_{1D})$ be a spanning subgraph of G in which $d_{1D}(S_i, S'_i) = d(S_i, S'_i)$ for each pair $(S_i, S'_i) \in D$. Let $v_j \in U' \iff \{v_j, v'_j\} \in E_{1D}$.

We need to show that U' is minimal if and only if $d(E_{1D})$ is minimal. Let us suppose that there is a non-empty subset $U_0 \subset U$ such that $|U_0| < |U'|$ and for $\forall i \in \{1, \dots, p\}$ $U_0 \cap S_i \neq \emptyset$. We can form a spanning subgraph $G_0 = (V, E_0)$ of G such that $\{v_j, v'_j\} \in E_0 \iff v_j \in U_0$ and for $i = 1, 2, \dots, p$ there exist exactly one path of length 3 from S_i to S'_i . Then

$$d(E_0) = 2p + |U_0| < 2p + |U'| \leq d(E_{1D}),$$

and this contradicts the fact that $d(E_{1D})$ is minimal.

Let us suppose that there is a spanning subgraph $G_0 = (V, E_0)$ of G such that $d_0(S_i, S'_i) = d(S_i, S'_i)$ for each pair $(S_i, S'_i) \in D$ and $d(E_0) < d(E_{1D})$. But $d(E_{1D}) = 2p + |U'|$ and $d(E_0) = p_1 + p_2 + u$ where $p_1 = \sum_{i=1}^p \deg(S_i)$ (resp. $p_2 = \sum_{i=1}^p \deg(S'_i)$) and u is the number of edges of the form $\{v_j, v'_j\}$ in E_0 . Since $p_1 + p_2 \geq 2p$ we obtain $u < |U'|$. Hence there is a set $U_0 = \{v_j \mid \{v_j, v'_j\} \in E_0\}$ with cardinality u for which $U_0 \cap S_i \neq \emptyset$ ($i = 1, \dots, p$). Then U' is not minimal.

It is an exercise from combinatorics to show that the above mentioned reduction is polynomial. $\square$

4 CONCLUSIONS

Several problems connected with the reduction of transport networks are studied in this paper. We show that the problems Admissible Lengthening of Routes and Admissible Lengthening of Important Routes – A General Version (for $q = 1$) are *NP*-hard problems. We also have proofs that the problems Limited Total Length of Complete Subnetwork and Admissible Lengthening of Important Routes – A General Version (for

$q > 1$) are *NP*-hard. However these proofs are too complicated. We plan to publish them in a further paper.

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Referee: