Planar Graphs and Graph Coloring Problem

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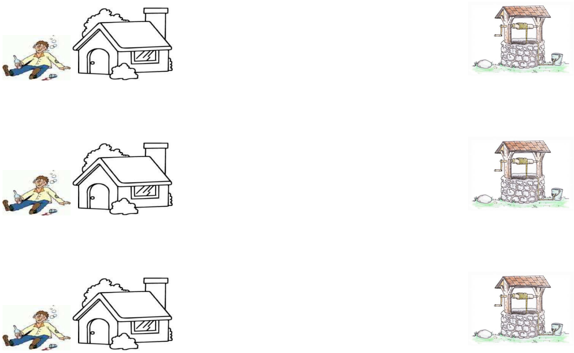
Three Cabins – Three Wells



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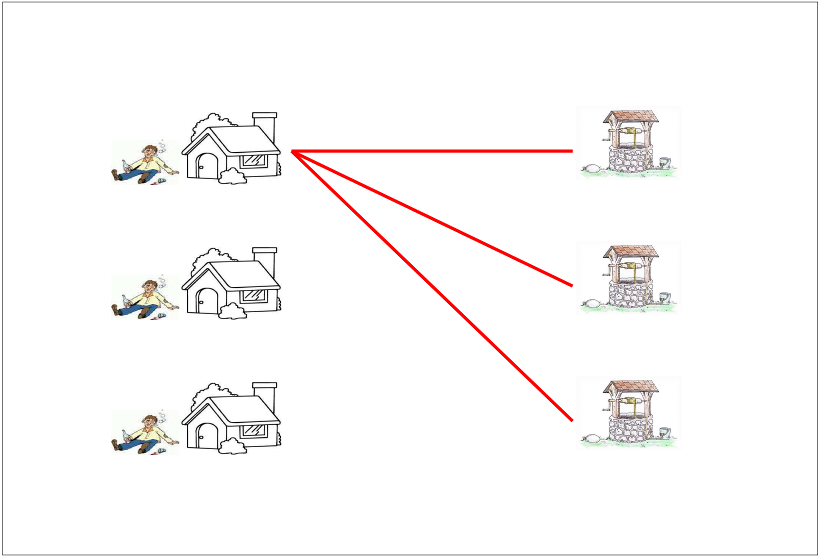

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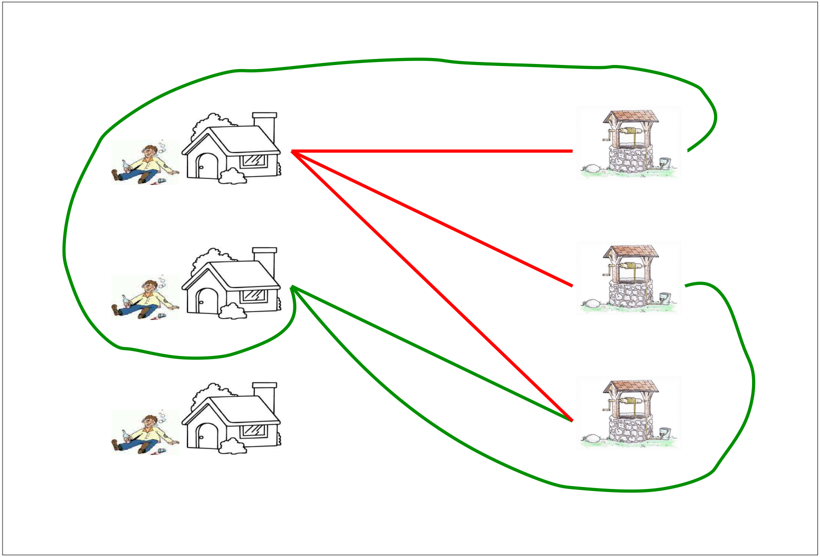

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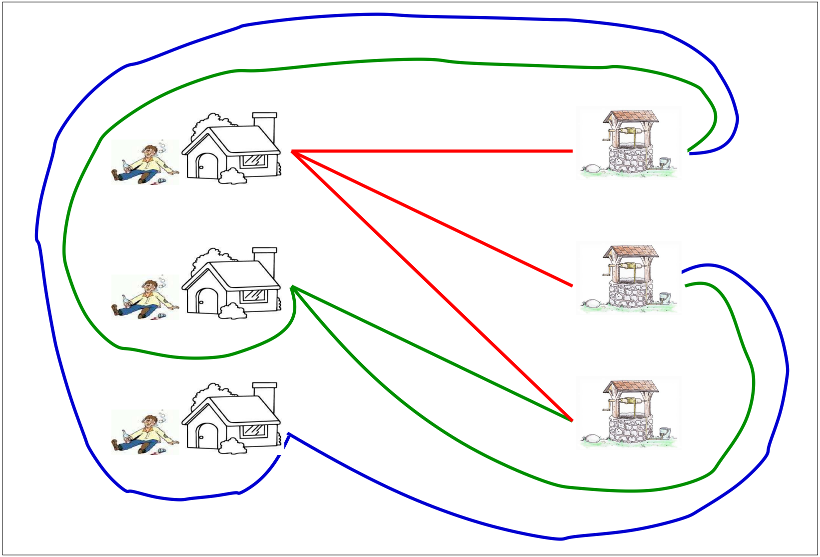
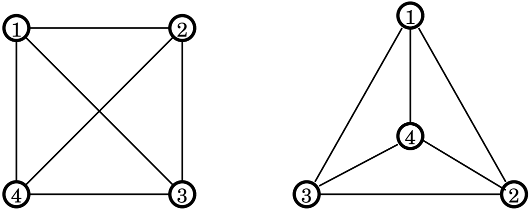


Definition

We will say that a diagram of a graph in an Euclidean plane is **a planar diagram**, *if its edges do not intersect nowhere exept vertices.*
A graph $G = (V, H)$ is **a planar graph** *if there exists a planar diagram of it.*



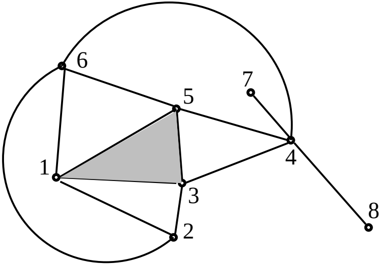
Obr.: Two diagrams of the same graph $G = (V, H)$,

where $V = \{1, 2, 3, 4\}$, H=$\{\{1, 2\},\{1, 3\},\{1, 4\}, \{2, 3\},\{2, 4\},\{3, 4\}\}$.

Face of a Planar Graph

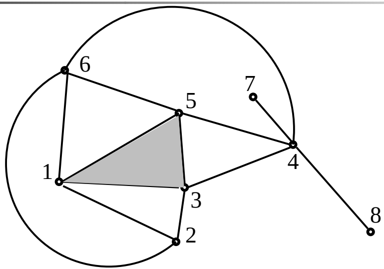
Definition

A face *of a planar diagram is the maximal part of the plane whose arbitrary two points can be joined by a continuous line which does not intersect any edge of that diagram.*



One face of a planar diagram.
Part of the plane bounded by edges $\{4, 5\}$, $\{5, 6\}$, $\{6, 4\}$ is a face.

Face of a Planar Graph



There are two types of faces – Exaxtly one face which is not bounded – this face is called **outer face**. Other faces are called **inner faces**.

Remark

Let us observe that vertices and edges of a diagram that determine a face create a „cycle".

There can exist also edges in a diagram that does not bound any face – such edges are $\{4, 7\}$, $\{4, 8\}$.

An edge is a border edge of a face if and only if it is contained at least in one cycle.

By extracting arbitrary edge of a cycle of a diagram the number of faces drops by 1.

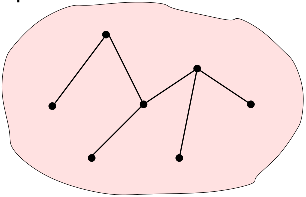
Theorem

Euler Polyhedral Equation *Let $G = (V, H)$ be a connected planar graph, let F be the set of faces in its planar diagram. The it holds:*

$$|F| = |H| - |V| + 2. \tag{1}$$

PROOF.

By mathematical induction by the number $|F|$ of faces of planar diagram. If $|F| = 1$ connected graph G does not contain a cycle – therefore G is a tree.

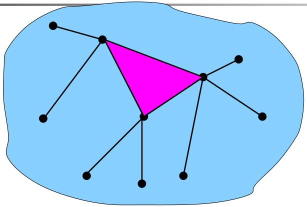


In a tree it holds $|H| = |V| - 1$.
Calculate: $|H| - |V| + 2 = (|V| - 1) - |V| + 2 = 1$.
For $|F| = 1$ we have

$$1 = |F| = |H| - |V| + 2.$$

Euler Polyhedral Equation

For $|F| = 2$



Be removing one edge h of a cycle we drop the number of faces.
The result is a planar connected graph $G' = (V, H')$, where

$$H' = H - \{h\}$$
$$|H'| = |H| - 1$$
with the following number of faces $|F'| = |F| - 1 = 2 - 1 = 1$

It holds for the case of one face:

$$1 = |F'| = |H'| - |V| + 2.$$

$$1 + 1 = |F'| + 1 = |H'| + 1 - |V| + 2$$
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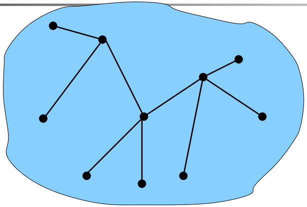

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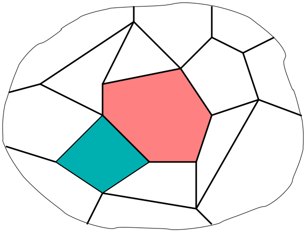
Let us have a graph with $|F| = |F'| + 1$ faces.



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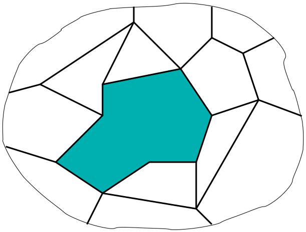

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Maximum of the Number of Edges in a Planar Graph

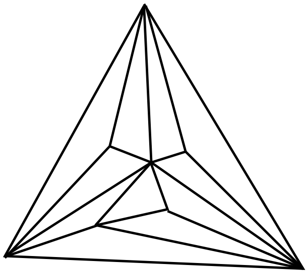
Theorem

Leth $G = (V, H)$ be a maximal planar graph with the vertex set V, where $|V| \geq 3$. Then

$$|H| = 3 \cdot |V| - 6. \tag{2}$$

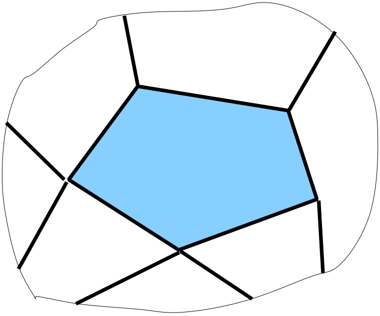
PROOF.

In a maximal planar graph with fixed vertex set V every face has to be a triangle – limited by 3 edges.



Maximum of the Number of Edges in a Planar Graph

If a inner face is not a triangle



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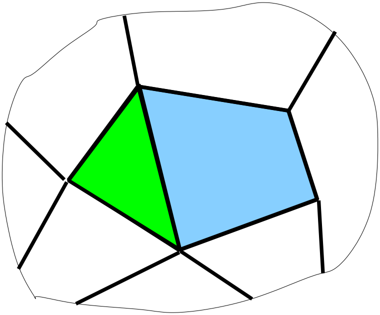

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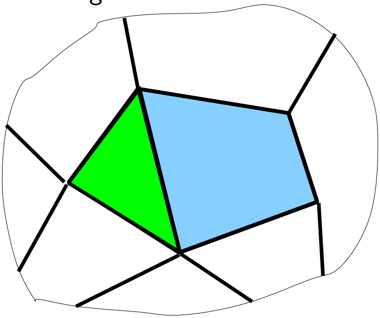

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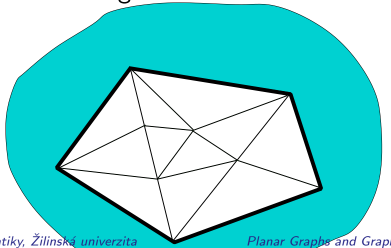

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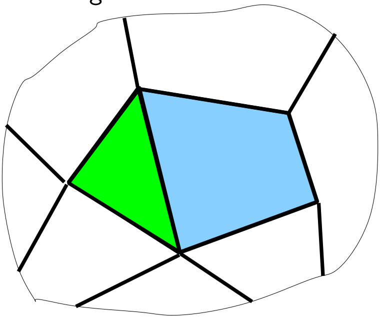

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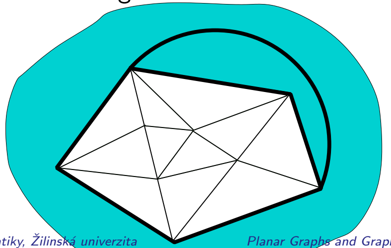

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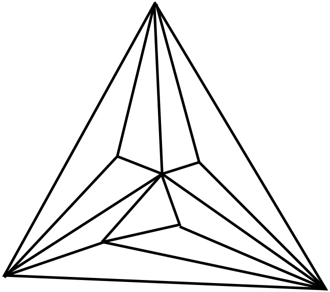

Every face is determined by 3 edges. If the triangles were disjoint (every edge only in one triangle) then we would need for them $3.|F|$ edges.

However, every edge is contained in exactly two faces, therefore the number of edges in a planar graph with a maximum number of edges is

$$|H| = \frac{3}{2} \cdot |F|$$

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Euler's polyhedral equation

$$|F| = |H| - |V| + 2$$

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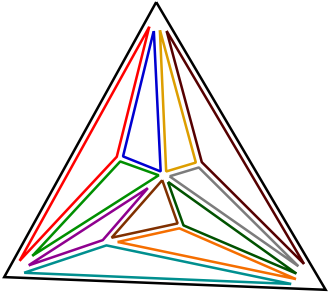

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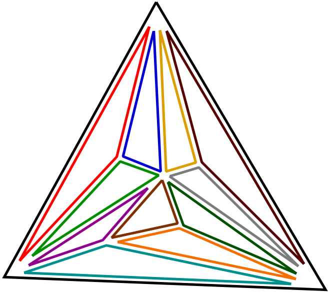

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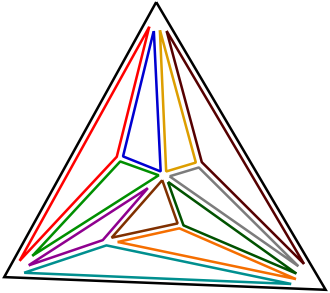

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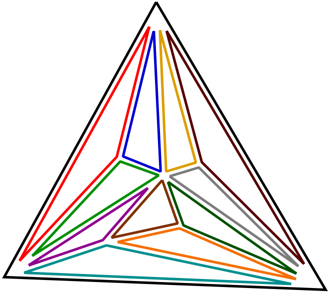

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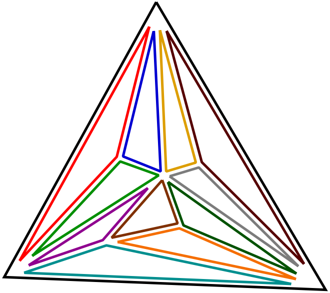

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Maximum of the Number of Edges in a Planar Graph

Corolary

It holds fopr every planar graph $G = (V, H)$, where $V \geq 3$:

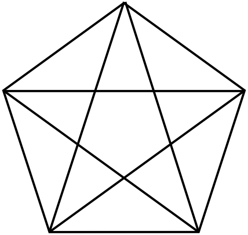
$$|H| \leq 3 \cdot |V| - 6.$$

Complete Graph K_5 is not Planar

Theorem

Complete graph K_5 with 5 vertices is not planar.

PROOF.



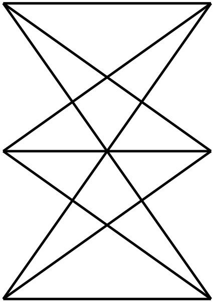
Complete graph K_5 has 5 vertices and $(5 \cdot 4)/2 = 10$ edges.
If it was planar, it could have at most $3 \cdot |V| - 6 = 3 \cdot 5 - 6 = 9$ edges.

Theorem

Complete bipartite graph $K_{3,3}$ is not planar.

PROOF.

Suppose that graph $K_{3,3}$ is planar.
Then its diagram does not contain a triangle – i. e. all its faces are squares or n-angles with $n \geq 4$.



Let diagram of graph $K_{3,3}$ has $|F|$ faces.
If border lines of a n-angles were disjoint we would need for them at least $4.|F|$ edges.

Since every edge in diagram is in two n-angles we nedd at least $4.|F|/2 = 2.|F|$ edges, i. e. $|H| \geq 2|F|$

$$|H| \geq \quad 2 \cdot \underbrace{|F|}_{=|H|-|V|+2} = 2 \cdot |H| - 2 \cdot |V| + 2 \cdot 2$$

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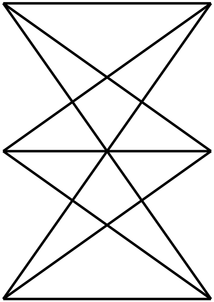

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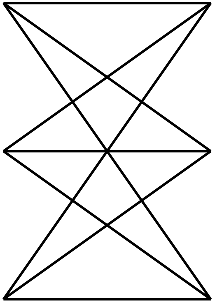

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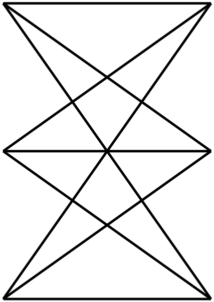

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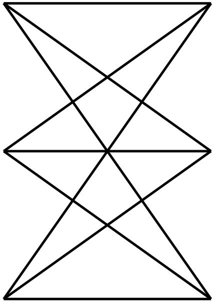

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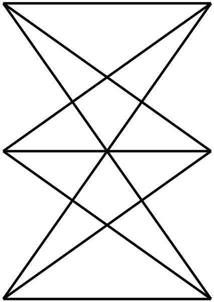

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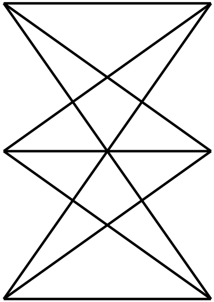

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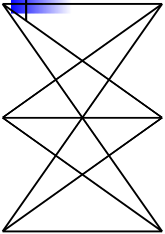
Since every edge in diagram is in two n-angles we nedd at least $4.|F|/2 = 2.|F|$ edges, i. e. $|H| \geq 2|F|$

$$\begin{aligned} |H| &\geq \quad 2 \cdot \underbrace{|F|}_{=|H|-|V|+2} = 2 \cdot |H| - 2 \cdot |V| + 2 \cdot 2 \\ -|H| &\geq -2 \cdot |V| + 4 \\ |H| &\leq \quad 2 \cdot |V| - 4 \end{aligned}$$

Complete Bipartite Graph $K_{3,3}$ is not planar



Graph $K_{3,3}$ has 9 edges. It has 6 vertices and his diagram does not contain a triangle.
If $K_{3,3}$ was a planar graph it could have at most $2.6 - 4 = 8$ edges – therefore $K_{3,3}$ can not be planar.

Definition

We will say that the graph $G' = (V', H')$ originated from the graph $G = (V, H)$ **by subdividing the edge** *$h \in H$, if*

$$V' = V \cup \{x\} \quad \textit{where } x \notin V,$$

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We will say that graphs $G = (V, H)$, $G' = (V', H')$ are **homeomorphic**, *if they are isomorphic or if it is possible to get from them a pair of isomorphic graphs by a finite sequence of subdividings of edges of both of them.*

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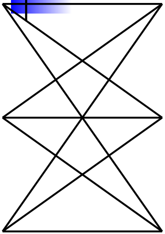

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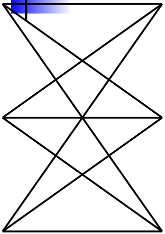

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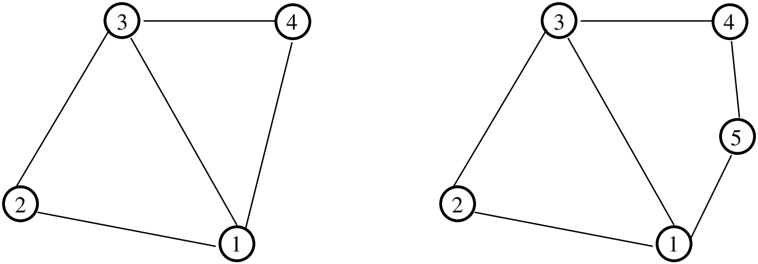
Homeomorphic Graphs



a) Graph G b) Graph $\overline{G}$

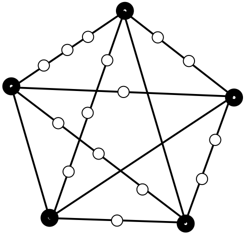
Homeomorphic Graphs.

Graph $\overline{G}$ originated from graph G by subdividing of edge $\{1, 4\}$.

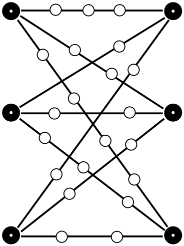
Kuratowski's Theorem

Theorem

Kuratowski. *Graph G is planar if and only if it does not contain a subgraph that is homeomorphic K_5 or $K_{3,3}$.*



a) Graph homeomorphic to K_5



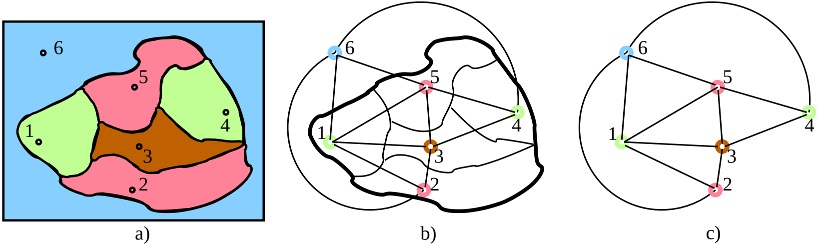
b) Graph homeomorphic to $K_{3,3}$

Two prototypes of non planar graphs.

These two types of graphs are known in literature as **Kuratowski's graphs.**

Geographical Maps Coloring Problem

Geographical Map Coloring Problem:
To color states of a political map by minimum number of colors so that no two neighbouring states (i.e. states with common border) are colored with the same color.



Graph model for Geographical Map Coloring Problem.
a) assign one vertex (6) to sea and a vertex to every state,
b) "join" two vertices corresponding to neighbouring states an edge,
c) diagram of resulting graph.

We have just transformed the Geographical Map Coloring Problem to the following problem:
To color vertices of a graph by minimum number of colors so that no two adjacent vertices are of the same color.

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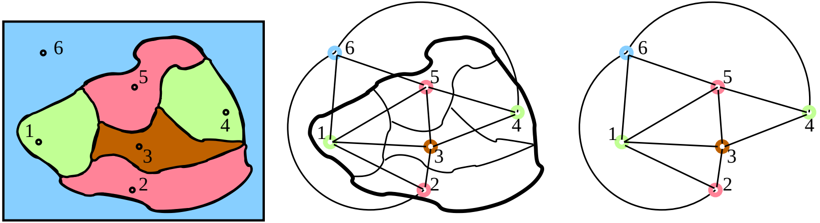

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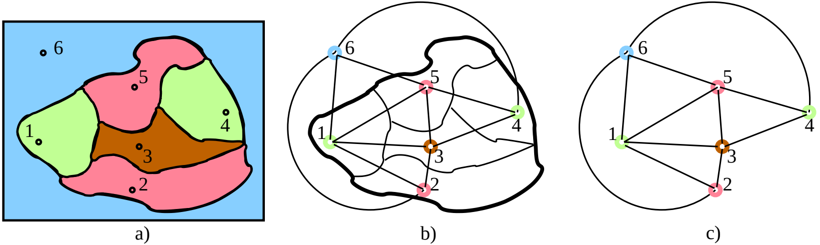

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Vertex Coloring Problem and Cromatic number of a Graph

Definition

A vertex coloring *of a graph* $G = (V, H)$ *is a function which assignes a color to every vertex* $v \in V$*.*

For every positive integer k*, a* **vertex** k**-coloring** *is a vertex coloring that uses exactly* k *colors.*

A proper vertex coloring *of a graph is a vertex coloring such that every two adjacent vertices are of different colors.*

A graph $G = (V, H)$ *is called* **vertex** k**-colorable** *if it has a proper vertex* k*-coloring.*

The chromatic number of a graph G*, denoted by* $\chi(G)$*, is the minimum number* k *such that* G *is* k*-colorable.*

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Heuristics for Vertex Coloring Preoblem

Theorem

Vertex Coloring Problem is NP-hard.

Algorithm

Sequential vertex coloring algorithm.

- **Step 1.** *Let* $\mathcal{P} = v_1, v_2, \ldots, v_n$ *be arbitrary sequence of vertices of graph* $G = (V, H)$.
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Sequential vertex coloring algorithm needs for its vertex coloring at most

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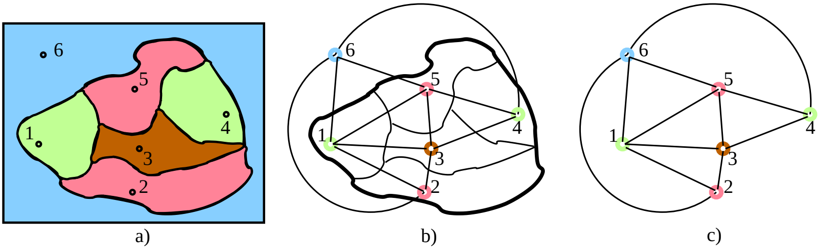

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Appel, Haken, 1976. *Every planar graph is 4-colorable.*

Remark

- *It was proven in the late 19th century (Heawood 1890) that every planar graph is 5-colorable.*

  However, no one could find a planar graph G with chromatic number $\chi(G) = 5$.

- *Theorem on 4-colorability was the first major theorem to be proved using a computer.*

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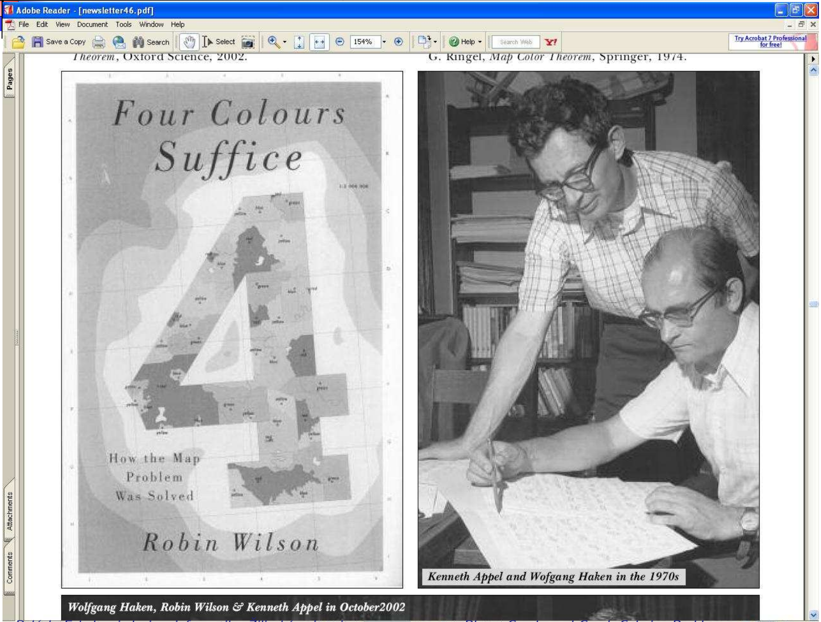

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Theorem, Oxford Science, 2002.
G. Ringel, Map Color Theorem, Springer, 1974.
Four Colours Suffice
How the Map Problem Was Solved
Robin Wilson
Kenneth Appel and Wofgang Haken in the 1970s
Wolfgang Haken, Robin Wilson & Kenneth Appel in October2002
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- **Step 1.** *Sort all vertices of graph* $G = (V, H)$ *into the sequence* $\mathcal{P} = (v_1, v_2, \dots, v_n)$ *by vertex degree in non ascending order.*
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LDF (Largest Degree First) Vertex Coloring Algorithm

The following algorithm is very similar to sequential coloring algorithm. The only difference is that sequential coloring assign colors to vertices in advance defined order while this algorithm chooses during course of algorithm which vertex will be colored next. iteľná farba.

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- Minimization of the number of shopping bags
- Minimization of the number of phases on traffic signal crossing
- Scheduling of school courses into minimal number of time slots
- Minimization of the number of bus stops on a bus station
- Etc. ....

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