CONNECTIONS BETWEEN THE GRAPH ISOMORPHISM AND THE NUMBER OF WALKS IN GRAPHS

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Graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ are isomorphic ($G \cong H$), if there is a bijection $f : V_G \longrightarrow V_H$ such, that for $\forall u, v \in V_G$: $\{u, v\} \in E_G \iff \{f(u), f(v)\} \in E_H$.

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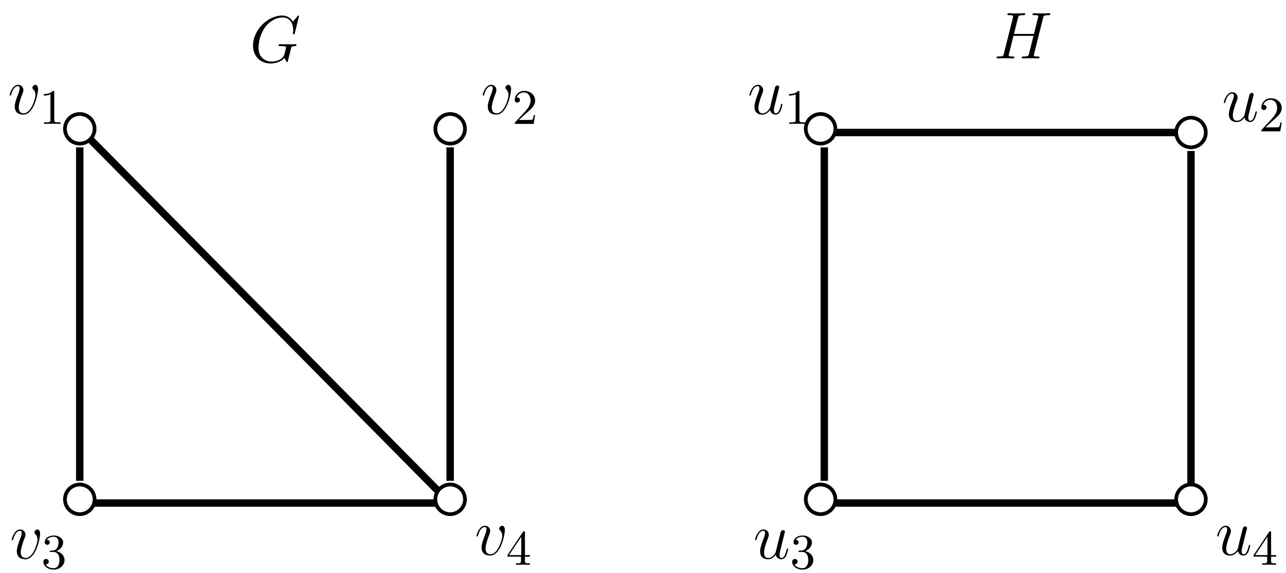
Let the graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ with adjacency matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ be given (where $V(G) = \{v_1, \ldots, v_n\}$, $V(H) = \{u_1, \ldots, u_n\}$).

Let $a_{ij}^{(k)}$ be the number of walks of lenght k from v_i to v_j.

$a_{ij}^{(k)} = \sum_{s=1}^{n} l_{i,s} \cdot l_{j,s} \cdot \lambda_s^k$

w-algorithm



$v_i \longrightarrow V_i$, $u_i \longrightarrow U_i$

$V_1 = \{s_{11}, s_{12}, s_{13}, s_{14}\}$, $U_1 = \{r_{11}, r_{12}, r_{13}, r_{14}\} = U_2 = U_3 = U_4$

$s_{11} = (1, 0, 2, 2, \ldots)$, $s_{12} = (0, 0, 1, 1, \ldots)$,

$s_{13} = (0, 1, 1, 3, \ldots)$, $s_{14} = (0, 1, 1, 4, \ldots)$,

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The Walk Algorithm

$v_i \longrightarrow V_i = \{s_{i1}, \ldots, s_{in}\}$, $u_i \longrightarrow U_i = \{r_{i1}, \ldots, r_{in}\}$, where $s_{ij} = \{a_{ij}^{(k)}\}_{k=0}^{q}$ and $r_{ij} = \{b_{ij}^{(k)}\}_{k=0}^{q}$ (q is the number of distinct non-zero eigenvalues of graphs G and H)

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- generating sets $V_1, \ldots, V_n$ and $U_1, \ldots, U_n$.
- finding a bijection $f : V(G_1) \longrightarrow V(G_2)$ for which the following condition holds: if $u_j = f(v_i)$ then $V_i = U_j$

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The complexity of the algorithm is $O(n^5)$.

Results

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If two graphs G and H are not cospectral then $G \not\propto_w H$.

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Theorem 3
If G and H have different angle matrices then $G \nsim_w H$.

Angle matrix

$$\begin{array}{c|ccc} & e_1 & \dots & e_n \\ \hline \varepsilon(\mu_1) & \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon(\mu_k) & \alpha_{k1} & \dots & \alpha_{kn} \end{array}$$

$$\alpha_{i,j} = \cos \measuredangle(\varepsilon(\mu_i), e_j)$$

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Definition.
A strongly regular graph (SRG) with parameters (n, k, b, c) is a regular graph with n vertices, every vertex has degree k, every pair of adjacent vertices has b common neighbours and every pair of distinct nonadjacent vertices has c common neighbours.

The Ulam graph reconstruction

Conjecture:
If G and H are counterexamples for Ulam reconstruction conjecture then $G \sim_w H$.

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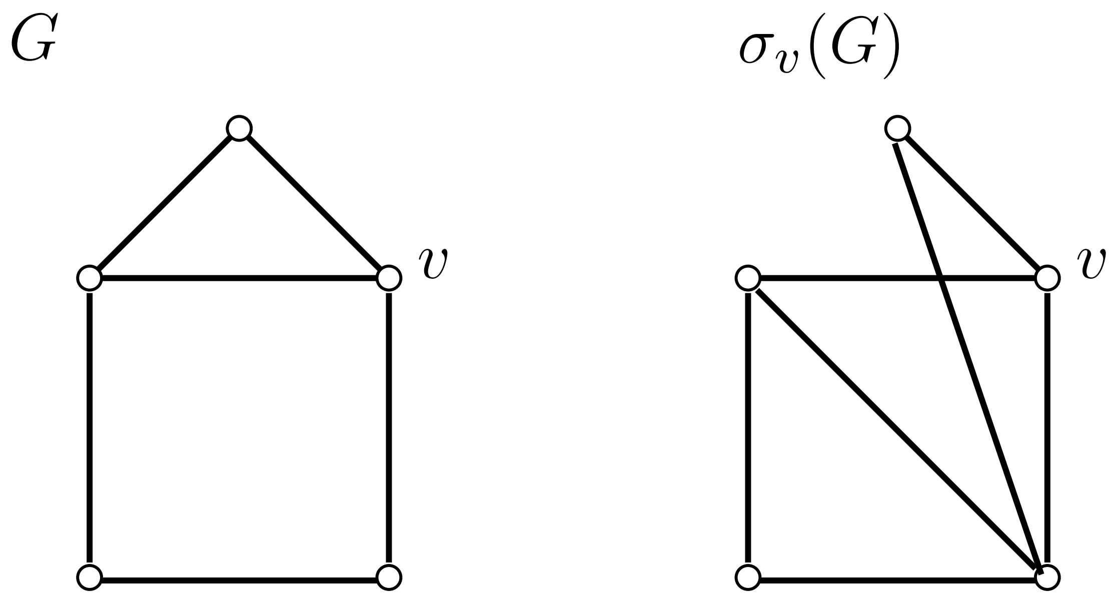
It is known that $w_i^{(k)} = \sum_{j=1}^{n} a_{ij}^{(k)}$ and $a_{ii}^{(k)}$ are reconstructible for $\forall i \in \{1, \ldots, n\}$.

Is $a_{ij}^{(k)}$ reconstructible for $i \neq j$?

Improvement of the w-algorithm

Local complement $\sigma_v(G)$ of G at vertex v (A. Bouchet):



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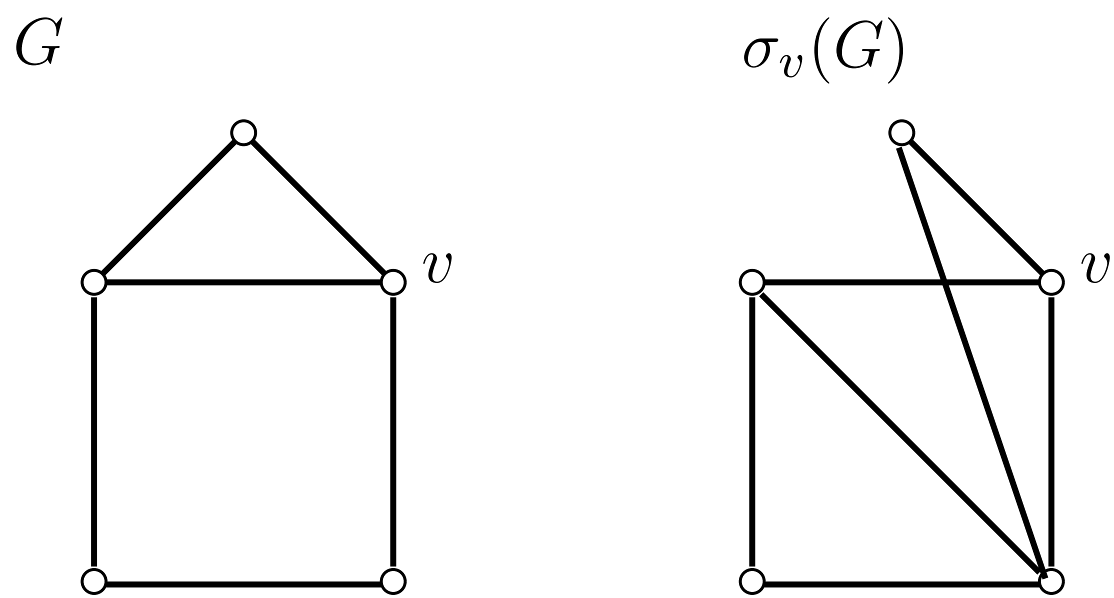

$G \longrightarrow \sigma_{v_1}(G), \ldots, \sigma_{v_n}(G)$
$H \longrightarrow \sigma_{u_1}(H), \ldots, \sigma_{u_n}(H)$
The complexity of the improved algorithm is $O(n^7)$.