Inverse Optimization for Bus Scheduling Problems

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About what ?

We study a nontraditional model for the real-world situation which can occur in bus scheduling problem.

It is motivated by the question: What minimal shortenings of the driving time should be made in some sections of an existing bus schedule (on a given transport network) to obtain optimal schedule?

- We show how this problem can be solved via an inverse optimization using linear programming (LP).
- Our approach is based on a nontraditional model for the computation the distance matrix in the transport network.
- We present computational experiments with the instance of public bus service for slovakian city Martin.

Inverse LP problem (Xu & Xu, 2013)

The inverse models describe the situation, when is known a feasible solution $\mathbf{x}^0$, which is not optimal with respect to the objective function. Unlike conventional optimization methods are not looking for the optimal solution to this problem but the objective function is looking for the smallest possible adjustment coefficients of this after which the feasible solution be the optimal solution.

Given a general LP problem (LP):: $\min\{\mathbf{cx} | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{c} \in R^n, \mathbf{A} \in R^{m \times n}$. Let $\mathbf{x}_0$ be a feasible solution, we need to change the vector $\mathbf{c}$ as least as possible and let $\mathbf{x}_0$ become an optimal solution of the adjusted LP. If we define

$$F(\mathbf{x}^0) = \{\bar{\mathbf{c}} \in R^n | \min\{\bar{\mathbf{c}}\mathbf{x} | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} = \bar{\mathbf{c}}\mathbf{x}^0\},$$

then the inverse problem of LP can be expressed as

$$\min\{\| \mathbf{c} - \bar{\mathbf{c}} \| \,|\, \bar{\mathbf{c}} \in F(\mathbf{x}^0)\}.$$

LP model for distance matrix

We model the transport network by a connected weighted digraph $G = (V, E, d)$ whose vertices V are nodes in the transport network formed by bus stops or crossroads and oriented edges E are evaluated with length $d(e), e \in E$ in minutes.

Denote a set $A = \{(i,j) | i \in V, j \in V, i \neq j\}$ and between its elements (i,j) we will search lengths of the shortest $i - j$ paths $z(i,j)$. Lengths of the shortest paths in network can be calculated through (DLP):

$$\sum_{(i,j)\in A} z(i,j) \to max$$

$$z(i,k) + z(k,j) \geq z(i,j), \quad \forall (i,k) \in E, \forall (k,j) \in A, \forall (i,j) \in A : \quad i \neq j \neq k \neq i,$$

$$z(i,j) \leq d(i,j), \qquad \forall (i,j) \in E,$$

$$z(i,j) \geq 0, \qquad \forall (i,j) \in A.$$

Lexicographic bus scheduling problem

Let be given the set of bus trips $T = \{t_1, t_2, \ldots, t_n\}$ on the considered bus network, which is modeled via connected time-weighted digraph $G = (V, E, d)$. The trip t_i is defined by ordered quadruple $t_i = (t_i^d, t_i^a, p_i^d, p_i^a), i \in N = \{1, 2, \ldots, n\}$, where

- t_i^d– departure time of the trip t_i; $t_i^d \in \langle 0, 1440 \rangle$,
- t_i^a– arrival time of the trip t_i; $t_i^a \in \langle 0, 1440 \rangle$,
- p_i^d– departure place of the trip t_i; $p_i^d \in V$,
- p_i^a– arrival place of the trip t_i; $p_i^a \in V$.

Let's have two trips t_i and t_j. We will say trip t_i precedes the trip t_j, and we will write $t_i \prec t_j$, if

$$t_i^a + d(p_i^a, p_j^d) \leq t_j^d,$$

where $d(p_i^a, p_j^d)$ is the minimum travel time from place p_i^a to place p_j^d in the bus network G.

A bus schedule is a set of running boards covering set given of trips T.

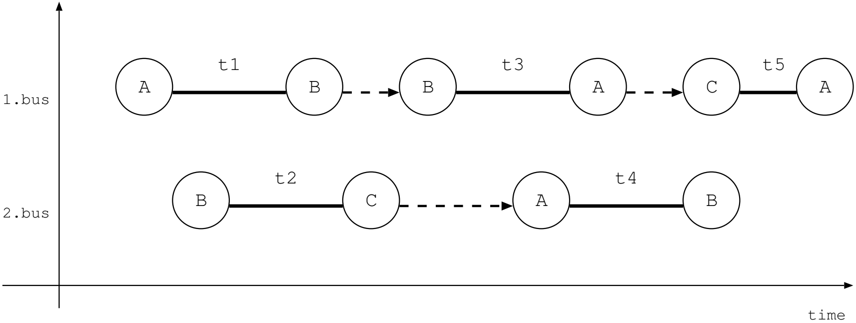

Figure 1: Bus schedule: $t_1 \prec t_3 \prec t_5, t_2 \prec t_4$.

Our goal is to find the bus schedule with minimum number of running boards $q_{\min}$ which has minimum total travel times between trips.

Let $x_{ij}, (i \in N, j \in N)$ be a decision variable,

$$x_{ij} = \begin{cases} 1 & \text{if } t_i \prec t_j \text{ or } t_i \text{ is a last trip and} \\ & t_j \text{ is a first trip in running boards,} \\ 0 & \text{otherwise,} \end{cases}$$

weighted by cost

$$c_{ij} = \begin{cases} d(p_i^a, p_j^d) & \text{if } t_i \prec t_j, \\ M & \text{otherwise,} \end{cases}$$

where M is a big penalization constant.

We can solve our problems the same way as known assignment problem (AP):

$$\sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij} \rightarrow \min \tag{1}$$

$$\sum_{j \in N} x_{ij} = 1 \quad \forall i \in N, \tag{2}$$

$$\sum_{i \in N} x_{ij} = 1 \quad \forall j \in N, \tag{3}$$

$$x_{ij} \geq 0 \quad \forall (i,j) \in N \times N. \tag{4}$$

Minimum number of running boards $q_{\min}$ is given by the number values M in value of object function

$$q_{\min} = \left\lfloor \frac{\sum_{i \in N} \sum_{j \in N} c_{ij} x_{ij}}{M} \right\rfloor .$$

For the inverse model we utilize the dual formulation of the assignment problem (DAP):

$$\sum_{i \in N} u_i + v_i \quad \rightarrow \quad \max \tag{5}$$

$$u_i + v_j \quad \leq \quad c_{ij} \qquad \forall (i,j) \in N \times N. \tag{6}$$

Let x_{ij}, u_i, v_j be variables of optimum solutions for problems AP and DAP. Then all $(i,j) \in N \times N$ must satisfy following complementarity conditions:

$$x_{ij} = 1 \Rightarrow u_i + v_j = c_{ij}, \tag{7}$$

$$x_{ij} = 0 \Rightarrow u_i + v_j \leq c_{ij}. \tag{8}$$

Inverse model for bus scheduling problem

The bus trips T given on the bus network are modeled by the time-weighted digraph $G = (V, E, d)$. But we suppose we do not know the exact time evaluating its edges, only its upper limit $d(i,j), (i,j) \in E$ is known.

Without any loss of generality, we will assume that the set of bus trips $T = \{(t_i^d, t_i^a, p_i^d, p_i^a) | i \in N\}$ is ordered by departure times:

$$t_1^d \leq t_2^d \leq \cdots \leq t_n^d.$$

We assume that the bus schedule is given in form of permutations ψ on the set of the trip-indices N, connections with following interpretation: Let be $j = \psi(i)$, then either

- $t_i \prec t_j$ and exist empty trip from trip t_i to trip t_j in some running board ,where $i < j$, or
- t_i is last trip and t_j is the first trip in running boards (or even the same running board), where $i \geq j$.

Our main goal is to find the smallest maximum reduction δ length of edges of the bus network, after which ψ becomes optimal bus schedule.

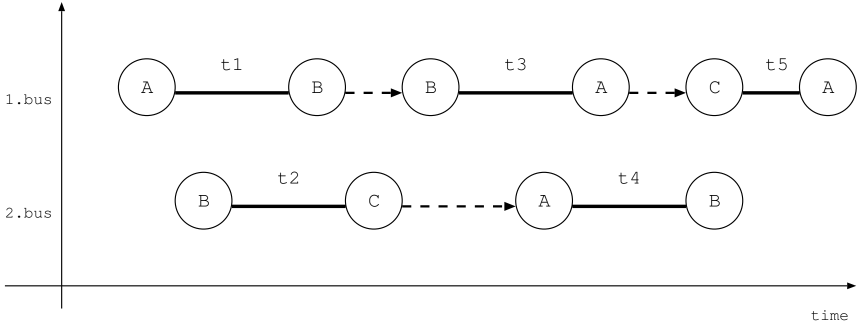

Figure 2: Bus schedule $\psi : \psi(1) = 3, \psi(2) = 4, \psi(3) = 5, \psi(4) = 2,$ $\psi(5) = 1.$

Next we use variables $z(i,j), u_i, v_j$ from previous models, new nonnegative variable δ and index sets A and B.

ILP model

$$\sum_{(i,j)\in A} (z(i,j) - \delta) \to max$$

$$z(i,k) + z(k,j) \geq z(i,j), \quad \forall (i,k) \in E, \forall (k,j) \in A, \forall (i,j) \in A : i \neq j \neq k \neq i,$$

$$d(i,j) - \delta \leq z(i,j) \leq d(i,j), \quad \forall (i,j) \in E,$$

$$u_i + v_{\psi(i)} = z(m_i^p, m_{\psi(i)}^o), \quad \forall i \in N : i < \psi(i),$$

$$u_i + v_{\psi(i)} = M, \quad \forall i \in N : i \geq \psi(i),$$

$$u_i + v_j \leq z(m_i^p, m_j^o), \quad \forall (i,j) \in B,$$

$$t_i^p + u_i + v_{\psi(i)} \leq t_{\psi(i)}^o, \quad \forall i \in N : i < \psi(i),$$

$$z(m_i^o, m_i^p) \leq t_i^p - t_i^o, \quad \forall i \in N,$$

$$z(i,i) = 0, \quad \forall i \in V,$$

$$z(i,j) \geq 0, \quad \forall (i,j) \in A,$$

$$u_i, v_i \geq 0, \quad \forall i \in N,$$

$$\delta \geq 0.$$

We applied the DLP, AP and ILP to a data sets of urban bus mass transit system for slovakian city Martin.
Our experiment has four phases:

1. We solved the DLP for digraph $G = (V, E, d)$, where
   - $V = \{p_i^d | i \in N\} \cup \{p_i^a | i \in N\}$,
   - $E = \{(p_i^d, p_i^a) | i \in N\}$,
   - $d(i,j) = \min_{k \in N}\{(t_k^a - t_k^d)\} | i = p_k^d, j = p_k^a\}$,

   with distance matrix $D = (d_{ij}), (i,j) \in A$ as the optimum solution.
2. We solve the AP with given set of bus trips T and distance matrix $D = (d_{ij})$.
3. We simulated 5 current bus schedules like this: In optimum solution X from 2. phase we random, step by step modify X so that we might receive permutations $\psi^1, \psi^2, \ldots, \psi^5$ representing feasible running boars.
4. We solved the ILP for 5 instances.

Experiments were conducted on HP XW6600 Workstation (8-core Xeon 3GHz, RAM 16GB) with OS Linux (Debian/jessie). We used Python-based tools and the Python interface to commercial solver Gurobi.

As expected execution time for $|T| = 726, |V| = 33, |E| = 98$ was very short, for the DLP 0.02 sec., for the AP 1.87 sec. and for the ILP 0.03 sec.

Table 1: Computational characteristics for the ILP.

Instances	1	2	3	4	5
δ (min)	16	16	16	16	15
Reduced edges	10	10	10	10	9
Objective	29084	28408	27196	23846	26644

We expect that the next calculations will lead to further refinement of our model. One of the possible refinement could be replacing the maximal value by which we can decrease the lenght of an edge by values dependent on the edge. It is also possible to demand fixed travel time.

Thank you for your attention...

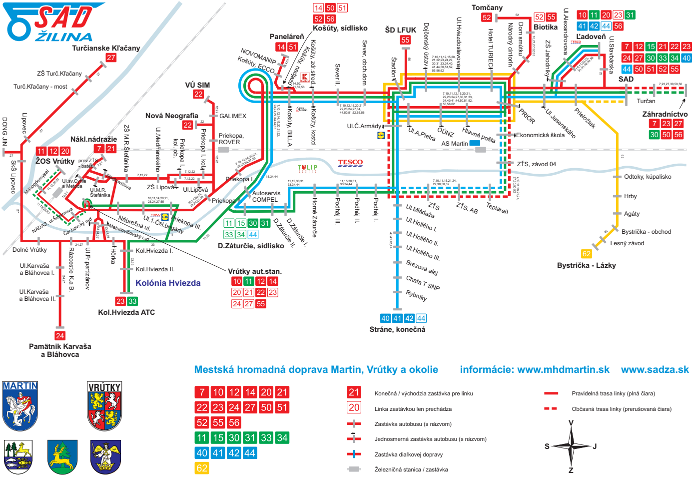

Figure 3: Bus network of city Martin.