Erlang ON/OFF Moduled Queueing Systems

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Abstract

We consider models with an Erlang ON/OFF moduled Poisson process ($EMPP_2$) of arrivals where service times of sources are distributed according to the Erlang distribution. We study a steady-state distribution of semi-Markov systems $EMPP_2/M/1/K$ and $EMPP_2/M/1/\infty$. Problem of overload analysis of the finite system is discussed. The queue model of the transport-inventory system is presented.

Keywords: Erlang ON/OFF moduled Poisson processes, overload analysis, $EMPP_2|M|1$ queueing, transport-inventory system

1 Introduction

Markov ON/OFF modulated Poisson process ($MMPP_2$) have been widely used to model different aspects of arrivals in communication and transport systems. Problem is that a variance of the service time of ON/OFF states of source are not exponential always. We consider models with an Erlang ON/OFF moduled Poisson process ($EMPP_2$) of arrivals where service times of source are distributed according to the Erlang distribution.

We study a steady-state distribution of the semi-Markov $EMPP_2/M/1K$ and $EMPP_2/M/1/\infty$ systems. A basic characteristics are derived from the corresponding continuous time Markov chain. Problem of overload analysis of the finite system is discussed. Results are applied in model of the transport-inventory system.

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2 Erlang ON/OFF moduled Poisson process

A *Markov modulated Poisson source* $MMPP$ [1] is represented by a couple (Q, Λ), the first matrix being the generator of the finite Markov chain $\{X(t), t \geq 0\}$ and the second being the diagonal matrix of the arrival intensities associated with each state. A special case and one of the most studied cases, is $MMPP$ with a two-state Markov chain noted $MMPP_2$.

An *Erlang ON/OFF moduled Poisson process* $EMPP_2$ have the ON/OFF states of source distributed by the Erlang $E_r(r\alpha)$ and $E_s(s\beta)$ distributions and customers are generated in ON phases of source according to a Poisson process with parameter λ. So the $EMPP_2$ process is the Markov modulated Poisson source based on a continuous-time irreducible Markov chain $\{X(t), t \geq 0\}$ with finite state space $M = \{1, 2, \ldots, r, r+1, \ldots, r+s\}$ and is represented by a couple of matrices $(\mathrm{Q}, \Lambda) = ((q_{ij})_{i,j \in M}), (\lambda_{ij})_{i,j \in M})$, where non-diagonal elements of matrices are

$$q_{ij} = \begin{cases} r\alpha & \text{if } 1 \leq i \leq r,\ j = i+1 \\ s\beta & \text{if } r < i < r+s,\ j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$\lambda_{ij} = \begin{cases} \lambda & \text{if } 1 \leq i \leq r,\ j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Note that r and s are positive integer numbers noted numbers of phases. Let $\boldsymbol{\phi}$ denote a row vector of the stationary distribution of the matrix Q. Also let $\boldsymbol{e}$ and $\boldsymbol{o}$ denote the column vector constituted of ones and zeros. From stationarity of source we have equations $\boldsymbol{o} = \boldsymbol{\psi}\mathrm{Q}$ and $\boldsymbol{\psi}\boldsymbol{e} = 1$ with steady-state solution

$$\boldsymbol{\phi} = \Big(\underbrace{\frac{\beta}{r(\alpha+\beta)}, \ldots, \frac{\beta}{r(\alpha+\beta)}}_{r}, \overbrace{\frac{\alpha}{s(\alpha+\beta)}, \ldots, \frac{\alpha}{s(\alpha+\beta)}}^{s} \Big). \tag{3}$$

The average rate of customers generated of the process $\bar{\lambda}$, the average $\bar{\tau}$ and variance $\bar{\tau}$ time of ON state (lifetime) of source are:

$$\bar{\lambda} = \boldsymbol{\phi}\Lambda\boldsymbol{e} = \frac{\lambda\beta}{\alpha+\beta}, \quad \bar{\tau} = \frac{1}{\alpha}, \quad \bar{\sigma} = \frac{1}{r\alpha^2}. \tag{4}$$

3 Queues of $\mathrm{EMPP_2/M/1}$ type

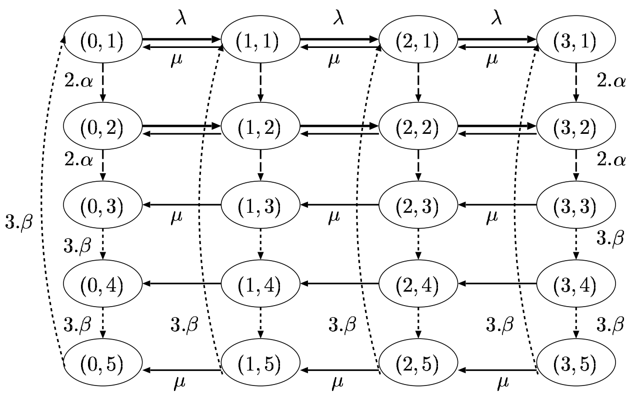

Figure 1: Transition graph of $EMPP_2/M/1/3$ queue

We consider semi-Markov infinite $EMPP_2/M/1/\infty$ and finite $EMPP_2/M/1/K$ queues. We assume that the time of service is distributed exponential with mean value $\frac{1}{\mu}$. An finite/infinite queue can be modelled by a homogeneous continuous-time Markov chain with a transition graph $G(\mathcal{S},\mathcal{H})$ or with a transition rate matrix of rates $\Omega = \big(\omega(x,x')\big)$. The countable state space is for finite case $\mathcal{S} = \{(l,i) : 0 \le l \le K, 1 \le i \le r+s\}$ and for infinite case $\mathcal{S} = \{(l,i) : l \ge 0, 1 \le i \le r+s\}$.

The first component, l, of the state descriptor $x = (l,i) \in \mathcal{S}$ denotes level - number of customers in system and and its second component, i, the ON/OFF phase of the source. The set of edges is $\mathcal{H} = \{(x,x') \in \mathcal{S}\times\mathcal{S} : q(x,x') > 0\}$ and transition rates are defined for levels $l = 0,1,2,\ldots,K$

$$\omega\big((l,i),(k,j)\big) = \begin{cases} \lambda & \text{if } 1 \le i \le r,\ k = l+1,\ j = i \\ r\alpha & \text{if } 1 \le i \le r,\ k = l,\ j = i+1 \\ s\beta & \text{if } r < i < r+s,\ k = l, \\ & \quad j = (i+1)mod(r+s)+1 \\ \mu & \text{if } 1 \le i \le r+s,\ 0 \le k = l-1,\ j = i \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Example of the transition graph for $EMPP_2/M/1/4$ queue with $E_2(2\alpha)$ distributed ON phase and $E_3(3\beta)$ distributed OFF phase is described on

the figure fig.1. First two horizontal levels of vertices represents ON phases and second three levels OFF phases of source. The vertical levels of vertices represents total number of customers in system.

For the finite queue we have following block-structured finite transition matrix of rates $\Omega_K \in \Re^{\mathcal{S}\times\mathcal{S}}$:

$$\Omega_K = \begin{pmatrix} \mathrm{A}_1 & \mathrm{B} & & & & \\ \mathrm{C} & \mathrm{A}_2 & \mathrm{B} & & & \\ & \mathrm{C} & \mathrm{A}_2 & \mathrm{B} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathrm{C} & \mathrm{A}_2 & \mathrm{B} \\ & & & & \mathrm{C} & \mathrm{A}_3 \end{pmatrix}$$

where submatrix B is diagonal with first r diagonal elements equal λ, submatrix C is diagonal with all elements equal μ and submatrices $\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3$ are semi-diagonal with negative diagonal. The order all block submatrices is $m = r+s$.

The solution $\boldsymbol{\pi}$ of the finite system $\mathbf{0} = \boldsymbol{\pi}\Omega_K$ and $\boldsymbol{\pi}\mathbf{1} = 1$ is steady-state distribution of the queue. Note that $\mathbf{0}$ and $\mathbf{1}$ denote column vector of ones and zero of length $|\mathcal{S}|$. This solution can be found numerically [4] from the singular value decomposition (SVD) of the matrix Ω_K.

4 Overload analysis

The $EMPP_2/M/1/\infty$ queue has block-tridiagonal infinitesimal matrix Ω

$$\Omega_\infty = \begin{pmatrix} \mathrm{A}_1 & \mathrm{B} & & & \\ \mathrm{C} & \mathrm{A}_2 & \mathrm{B} & & \\ & \mathrm{C} & \mathrm{A}_2 & \mathrm{B} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

and is known as a homogeneous continuous-time *quasi-birth-and-death Markov chain*. The fast algorithm for the steady-state distribution $\boldsymbol{\pi}$, if exists, has been developed by Latouche and Ramaswami [2]. This algorithm is based on the following resource: Let $\boldsymbol{\pi}$ be partitioned by levels into sub-vectors $\boldsymbol{\pi}_l, l \ge 0$ of length m so that $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots)$. Let $\boldsymbol{\pi}$ is invariant vector associated by a matrix Ω_K i.e.

$$\boldsymbol{\pi}\Omega_\infty = \mathbf{0},\ \ \boldsymbol{\pi}\mathbf{1} = 1 \tag{6}$$

Then $\boldsymbol{\pi}$ also satisfies the matrix-geometric property:

$$\boldsymbol{\pi}_{l+1} = \boldsymbol{\pi}_l \mathrm{R} \ \text{ for } l \geq 0, \tag{7}$$

where the matrix R of order m records the rate of visit to level $(l+1)$ per unit of time spent in level l. The matrix R is minimal nonnegative solution to the matrix quadratic solution

$$\mathrm{R} = \mathrm{B} + \mathrm{R}.\mathrm{A}_2 + \mathrm{R}^2.\mathrm{C}. \tag{8}$$

The case we are interested is such stabilised $EMPP_2/M/1/K$ queue that $\bar{\lambda}/\mu > 1$ i.e. *overload* case. Note that the buffer size $K < \infty$. The queue $PH/PH/1/K$ with very large buffer has been analysed by Alfa and Zhao [3]. This queue has a discrete time distribution of phases for inter-arrival and service times.

We use the rotation procedure presented in [3]. By rotating the matrix Ω_K through 180 to obtain

$$\Omega_K^* = \begin{pmatrix} \mathrm{A}_3^* & \mathrm{B}^* & & & & \\ \mathrm{C}^* & \mathrm{A}_2^* & \mathrm{B}^* & & & \\ & \mathrm{C}^* & \mathrm{A}_2^* & \mathrm{B}^* & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathrm{C}^* & \mathrm{A}_2^* & \mathrm{B}^* \\ & & & & \mathrm{C}^* & \mathrm{A}_1^* \end{pmatrix}$$

where for matrix $\mathrm{A} = (a_{ij})$ of order m has matrix $\mathrm{A}^* = (a_{i,j}^*)$ of order m elements $a_{i,j}^* = a_{n+1-i,n+1-j}$.

It is possible show, as in paper [3], that if vectors $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_0, \ldots, \boldsymbol{\pi}_K)$ and $\boldsymbol{\pi}^* = (\boldsymbol{\pi}_0^*, \boldsymbol{\pi}_0^*, \ldots, \boldsymbol{\pi}_K^*)$ are the probability invariant vector associated with matrices Ω_K and Ω_K^* than $\boldsymbol{\pi}^*$ is rotation of $\boldsymbol{\pi}$ i.e. for all level l is holds $\boldsymbol{\pi}_l^*(i) = \boldsymbol{\pi}_{K-l}(m+1-i), i = 1, 2, \ldots, m$.

We can now approximate $\boldsymbol{\pi}$ by an infinite state Markov chain with the following transition rate matrix

$$\Omega_\infty^* = \begin{pmatrix} \mathrm{A}_3^* & \mathrm{B}^* & & & \\ \mathrm{C}^* & \mathrm{A}_2^* & \mathrm{B}^* & & \\ & \mathrm{C}^* & \mathrm{A}_2^* & \mathrm{B}^* & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$

The infinite vector $\boldsymbol{\psi}$ which is the probability invariant vector associated with matrix Ω_∞^* has been establish by Latouche and Ramaswami [2] logarithmic reduction algorithm. Its traffic intensity is $\mu/\bar{\lambda} < 1$ and chain is

stable. In case with large buffer we can approximate $\boldsymbol{\pi} \approx (\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_K)$ and in case with small buffer by formulas

$$\boldsymbol{\pi}_l(i) \approx \frac{\boldsymbol{\psi}_l(i)}{\sum_{k=0}^{K} \sum_{j=1}^{m} \boldsymbol{\psi}_k(j)} \tag{9}$$

5 Transport-inventory system

Example for the $EMPP_2/M/1/K$ queue is a following transport-inventory system: Transport company supplies one manufacture with products in "ON phases" of days only. In "OFF phases" of days it is busy with other transport jobs. The manufacturing process is modelled as a single-server facility which produces units with a exponential service time distribution. Demands arrive in ON phase according to Poisson process. The mean and the variance of phases ON/OFF are given. The goal is to found a optimal size of storage when maintenance cost function – depending on transport cost, holding cost and a deficit cost – is minimised.

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