Contents

Basic integrals

$$\int dx = \int 1\, dx = x + c, \qquad \text{for } x \in R \quad \text{(i.e. } f(x) = 1,\ x \in R). \qquad [c = \text{constant}]$$

$$\int x^a\, dx = \frac{x^{a+1}}{a+1} + c, \qquad \text{for } a \in R,\ a \neq -1,\ x \in R - \{0\}.$$

$$\int \frac{dx}{x} = \ln|x| + c, \qquad \text{for } x \in R - \{0\}.$$

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + c, \qquad \text{for } f(x) \neq 0,\ x \in D(f).$$

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R, \qquad \text{special case: } \int e^x\, dx = e^x + c.$$

$$\int a^x\, dx = \frac{a^x}{\ln a} + c, \qquad \text{for } a > 0,\ a \neq 1,\ x \in R, \qquad \text{special case: } \int e^x\, dx = \frac{e^x}{\ln e} + c = e^x + c.$$

$$\int \sin ax\, dx = -\frac{\cos ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R, \qquad \text{special case: } \int \sin x\, dx = -\cos x + c.$$

$$\int \cos ax\, dx = \frac{\sin ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R, \qquad \text{special case: } \int \cos x\, dx = \sin x + c.$$

$$\int \frac{dx}{\cos^2 ax} = \frac{\operatorname{tg} ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R - \left\{(2k+1)\frac{\pi}{2}\,;\ k \in Z\right\}, \qquad \text{special case: } \int \frac{dx}{\cos^2 x} = \operatorname{tg} x + c.$$

$$\int \frac{dx}{\sin^2 ax} = -\frac{\operatorname{cotg} ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R - \{k\pi\,;\ k \in Z\}, \qquad \text{special case: } \int \frac{dx}{\sin^2 x} = -\operatorname{cotg} x + c.$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \begin{cases} \arcsin \dfrac{x}{|a|} + c, \\ -\arccos \dfrac{x}{|a|} + c, \end{cases} \qquad \text{for } a \in R,\ a \neq 0,\ x \in (-|a|\,;\ |a|).$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left|x + \sqrt{x^2 - a^2}\right| + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in (-\infty\,;\ -|a|) \cup (|a|\,;\ \infty).$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R.$$

$$\int \frac{dx}{x^2 + a^2} = \begin{cases} \dfrac{1}{a} \operatorname{arctg} \dfrac{x}{a} + c, \\ -\dfrac{1}{a} \operatorname{arccotg} \dfrac{x}{a} + c, \end{cases} \qquad \text{for } a \in R,\ a \neq 0,\ x \in R.$$

$$\int \frac{dx}{x^2 - a^2} = \int \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right] dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R - \{\pm a\}.$$

$$\int \sinh ax\, dx = \frac{\cosh ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R, \qquad \text{special case: } \int \sinh x\, dx = \cosh x + c.$$

$$\int \cosh ax\, dx = \frac{\sinh ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R, \qquad \text{special case: } \int \cosh x\, dx = \sinh x + c.$$

$$\int \frac{dx}{\cosh^2 ax} = \frac{\operatorname{tgh} ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R, \qquad \text{special case: } \int \frac{dx}{\cosh^2 x} = \operatorname{tgh} x + c.$$

$$\int \frac{dx}{\sinh^2 ax} = -\frac{\operatorname{cotgh} ax}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R - \{0\}, \qquad \text{special case: } \int \frac{dx}{\sinh^2 x} = -\operatorname{cotgh} x + c.$$

Basic integrals — Appendix

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-b^2x^2}} = \int \frac{\mathrm{d}x}{|b|\sqrt{\frac{a^2}{b^2}-x^2}} = \begin{cases} \dfrac{1}{|b|}\arcsin\dfrac{|b|\,x}{|a|}+c, \\[2ex] -\dfrac{1}{|b|}\arccos\dfrac{|b|\,x}{|a|}+c, \end{cases} \qquad \text{for } \; a,b\in R-\{0\},\ x\in\left(-\frac{|a|}{|b|}\,;\,\frac{|a|}{|b|}\right).$$

$$\int \frac{\mathrm{d}x}{\sqrt{b^2x^2-a^2}} = \int \frac{\mathrm{d}x}{|b|\sqrt{x^2-\frac{a^2}{b^2}}} = \frac{1}{|b|}\ln\left|x+\sqrt{x^2-\frac{a^2}{b^2}}\right|+c_1 = \frac{1}{|b|}\ln\left|\frac{|b|\,x+\sqrt{b^2x^2-a^2}}{|b|}\right|+c_1 =$$

$$= \frac{1}{|b|}\ln\left||b|\,x+\sqrt{b^2x^2-a^2}\right| - \frac{1}{|b|}\ln|b|+c_1 = \frac{1}{|b|}\ln\left||b|\,x+\sqrt{b^2x^2-a^2}\right|+c,$$

$$\text{for } \; a,b\in-\{0\},\ x\in\left(-\infty\,;\,-\frac{|a|}{|b|}\right)\cup\left(\frac{|a|}{|b|}\,;\,\infty\right).$$

$$\int \frac{\mathrm{d}x}{\sqrt{b^2x^2+a^2}} = \int \frac{\mathrm{d}x}{|b|\sqrt{x^2+\frac{a^2}{b^2}}} = \frac{1}{|b|}\ln\left(x+\sqrt{x^2+\frac{a^2}{b^2}}\right)+c_1 = \frac{1}{|b|}\ln\left(\frac{|b|\,x+\sqrt{b^2x^2+a^2}}{|b|}\right)+c_1 =$$

$$= \frac{1}{|b|}\ln\left(|b|\,x+\sqrt{b^2x^2+a^2}\right) - \frac{1}{|b|}\ln|b|+c_1 = \frac{1}{|b|}\ln\left(|b|\,x+\sqrt{b^2x^2+a^2}\right)+c, \qquad \text{for } \; a,b\in R-\{0\},\ x\in R.$$

$$\int \frac{\mathrm{d}x}{b^2x^2+a^2} = \int \frac{\mathrm{d}x}{b^2\left(x^2+\frac{a^2}{b^2}\right)} = \begin{cases} \dfrac{1}{b^2}\dfrac{b}{a}\operatorname{arctg}\dfrac{bx}{a}+c = \dfrac{1}{ab}\operatorname{arctg}\dfrac{bx}{a}+c, \\[2ex] -\dfrac{1}{b^2}\dfrac{b}{a}\operatorname{arccotg}\dfrac{bx}{a}+c = -\dfrac{1}{ab}\operatorname{arccotg}\dfrac{bx}{a}+c, \end{cases} \qquad \text{for } \; a,b\in R-\{0\},\ x\in R.$$

$$\int \frac{\mathrm{d}x}{b^2x^2-a^2} = \int \frac{\mathrm{d}x}{b^2\left(x^2-\frac{a^2}{b^2}\right)} = \frac{1}{b^2}\frac{b}{2a}\ln\left|\frac{x-\frac{a}{b}}{x+\frac{a}{b}}\right|+c = \frac{1}{2ab}\ln\left|\frac{bx-a}{bx+a}\right|+c, \qquad \text{for } \; a,b\in R-\{0\},\ x\in R-\left\{\pm\frac{a}{b}\right\}.$$

$$\int \frac{\mathrm{d}x}{a^2-b^2x^2} = -\int \frac{\mathrm{d}x}{b^2x^2-a^2} = -\frac{1}{2ab}\ln\left|\frac{bx-a}{bx+a}\right|+c = \frac{1}{2ab}\ln\left|\frac{bx+a}{bx-a}\right|+c, \qquad \text{for } \; a,b\in R-\{0\},\ x\in R-\left\{\pm\frac{a}{b}\right\}.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+qx+r}} = \begin{cases} \ln\left(2x+q+2\sqrt{x^2+qx+r}\right)+c, & \text{for } q,r\in R,\ 4r-q^2>0,\ x\in R, \\[2ex] \ln|2x+q|+c, & \text{for } q,r\in R,\ 4r-q^2=0,\ x\in R-\left\{-\frac{q}{2}\right\}, \\[2ex] \ln\left|2x+q+2\sqrt{x^2+qx+r}\right|+c, & \text{for } q,r\in R,\ 4r-q^2<0,\ x\in R,\ x^2+qx+r>0. \end{cases}$$

$$\boxed{x^2+qx+r=\left(x^2+\frac{q}{2}\right)^2+r-\frac{q^2}{4}=\left(x^2+\frac{q}{2}\right)^2+\frac{4r-q^2}{4}, \qquad x+\frac{q}{2}+\sqrt{x^2+qx+r}=\frac{2x+q+2\sqrt{x^2+qx+r}}{2}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{px^2+qx+r}} = \begin{cases} \frac{1}{\sqrt{p}}\ln\left(2px+q+2\sqrt{p}\sqrt{px^2+qx+r}\right)+c, & \text{for } p,q,r\in R,\ p>0,\ 4pr-q^2>0,\ x\in R, \\[2ex] \frac{1}{\sqrt{p}}\ln|2px+q|+c, & \text{for } p,q,r\in R,\ p>0,\ 4pr-q^2=0,\ x\in R-\left\{-\frac{q}{2p}\right\}, \\[2ex] \frac{1}{\sqrt{p}}\ln\left|2px+q+2\sqrt{p}\sqrt{px^2+qx+r}\right|+c, & \\ & \text{for } p,q,r\in R,\quad p>0,\quad 4pr-q^2<0,\quad x\in R,\quad px^2+qx+r>0, \\[2ex] \begin{cases} -\frac{1}{\sqrt{-p}}\arcsin\frac{2px+q}{\sqrt{q^2-4pr}}+c, \\ \\ \frac{1}{\sqrt{-p}}\arccos\frac{2px+q}{\sqrt{q^2-4pr}}+c, \end{cases} & \text{for } p,q,r\in R,\quad p<0,\quad 4pr-q^2<0,\quad x\in R,\quad px^2+qx+r>0. \end{cases}$$

$$\boxed{\begin{array}{l} px^2+qx+r=p\left(x^2+\frac{q}{p}x+\frac{r}{p}\right)=p\left[\left(x+\frac{q}{2p}\right)^2+\frac{r}{p}-\frac{q^2}{4p^2}\right]=p\left(x+\frac{q}{2p}\right)^2+\frac{4pr-q^2}{4p}=p\left(x+\frac{q}{2p}\right)^2-\frac{q^2-4pr}{4p} \\[2ex] p>0\colon \qquad \sqrt{p}\left(x+\frac{q}{2p}\right)+\sqrt{px^2+qx+r}=x\sqrt{p}+\frac{q}{2\sqrt{p}}+\sqrt{px^2+qx+r}=\frac{2px+q+2\sqrt{p}\sqrt{px^2+qx+r}}{2\sqrt{p}} \\[2ex] p<0\colon \quad p\left(x+\frac{q}{2p}\right)^2-\frac{q^2-4pr}{4p}=\frac{q^2-4pr}{-4p}-(-p)\left(x+\frac{q}{2p}\right)^2, \qquad \frac{\sqrt{-p}\left(x+\frac{q}{2p}\right)}{\sqrt{\frac{q^2-4pr}{-4p}}}=\frac{-2p\left(x+\frac{q}{2p}\right)}{\sqrt{q^2-4pr}}=-\frac{2px+q}{\sqrt{q^2-4pr}} \end{array}}$$

Substitution

Theorem.
$f(x)\in C(I),\ x=\varphi(t),\ t\in J,\ \varphi(I)\subset I,\ \varphi'(t)\in C(J) \implies \forall t\in J\colon \int f(x)\,\mathrm{d}x = \int f[\varphi(t)]\,\mathrm{d}\varphi(t) = \int f[\varphi(t)]\varphi'(t)\,\mathrm{d}t.$

$$\int \sqrt{1-x^2}\,\mathrm{d}x = \boxed{\begin{array}{c} x=\sin t, \quad t=\arcsin x \implies \mathrm{d}x=\cos t\,\mathrm{d}t \\ \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t, \quad x\in(-1\,;\,1) \Rightarrow t\in\left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right)\end{array}} = \int \cos^2 t\,\mathrm{d}t =$$

$$= \int \frac{1+\cos 2t}{2}\,\mathrm{d}t = \frac{t}{2}+\frac{\sin 2t}{4}+c = \frac{t}{2}+\frac{2\sin t\cos t}{4}+c = \frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+c, \qquad \text{for } x\in(-1\,;\,1).$$

$$\int \sin^3 t\cos t\,\mathrm{d}t = \boxed{\begin{array}{c} x=\sin t \\ \mathrm{d}x=\cos t\,\mathrm{d}t\end{array}} = \int x^3\,\mathrm{d}x = \frac{x^4}{4}+c = \frac{\sin^4 t}{4}+c, \qquad \text{for } t\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt[3]{x+1}} = \int \frac{\mathrm{d}x}{\left(\sqrt[6]{x+1}\right)^3+\left(\sqrt[6]{x+1}\right)^2} = \boxed{\begin{array}{c} \sqrt[6]{x+1}=t \\ x+1=t^6 \\ \mathrm{d}x=6t^5\,\mathrm{d}t\end{array}} = \int \frac{6t^5\,\mathrm{d}t}{t^3+t^2} = 6\int \frac{t^3\,\mathrm{d}t}{t+1} =$$

$$= 6\int \frac{t^3+t^2-t^2-t+t+1-1}{t+1}\,\mathrm{d}t = 6\int \left[\frac{t^3+t^2}{t+1}-\frac{t^2+t}{t+1}+\frac{t+1}{t+1}-\frac{1}{t+1}\right]\mathrm{d}t =$$

$$= 6\int \left[t^2-t+1-\frac{1}{t+1}\right]\mathrm{d}t = 6\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right]+c =$$

$$= 2\left(\sqrt[6]{x+1}\right)^3-3\left(\sqrt[6]{x+1}\right)^2+6\sqrt[6]{x+1}-6\ln\left|1+\sqrt[6]{x+1}\right|+c =$$

$$= 2\sqrt{x+1}-3\sqrt[3]{x+1}+6\sqrt[6]{x+1}-6\ln\left|1+\sqrt[6]{x+1}\right|+c, \qquad \text{for } x>-1.$$

$$\int \frac{\mathrm{d}x}{\sqrt{5+4\,\mathrm{e}^x}} = \boxed{\begin{array}{c} \mathrm{e}^x = t,\ t>0 \\ x=\ln t,\ \mathrm{d}x = \frac{\mathrm{d}t}{t} \end{array}} = \int \frac{\mathrm{d}t}{t\sqrt{5+4t}} = \boxed{\begin{array}{c} 4t=u^2 \\ 4\,\mathrm{d}t = 2u\,\mathrm{d}u \end{array}} = \int \frac{4}{u^2\sqrt{5+u^2}}\,\frac{u\,\mathrm{d}u}{2} = 2\int \frac{\mathrm{d}u}{u\sqrt{5+u^2}} =$$

$$= \boxed{\begin{array}{c} u=\frac{\sqrt{5}}{v} \implies \mathrm{d}u = -\frac{\sqrt{5}\,\mathrm{d}v}{v^2}, \quad u>0 \\ \sqrt{5+u^2} = \sqrt{5+\frac{5}{v^2}} = \frac{\sqrt{5}}{v}\sqrt{v^2+1}, \quad v>0 \end{array}} = 2\int \frac{v}{\sqrt{5}}\,\frac{v}{\sqrt{5}\sqrt{v^2+1}}\,\frac{-\sqrt{5}\,\mathrm{d}v}{v^2} = -\frac{2}{\sqrt{5}}\int \frac{\mathrm{d}v}{\sqrt{v^2+1}} =$$

$$= -\frac{2}{\sqrt{5}}\ln\left(v+\sqrt{v^2+1}\right)+c_1 = -\frac{2}{\sqrt{5}}\ln\left(\frac{\sqrt{5}}{u}+\sqrt{\frac{5}{u^2}+1}\right)+c_1 = -\frac{2}{\sqrt{5}}\ln\left(\frac{\sqrt{5}}{u}+\frac{\sqrt{5+u^2}}{u}\right)+c_1 =$$

$$= -\frac{2}{\sqrt{5}}\ln\frac{\sqrt{5}+\sqrt{5+u^2}}{u}+c_1 = \frac{2}{\sqrt{5}}\left[\ln u - \ln\left(\sqrt{5}+\sqrt{5+u^2}\right)\right]+c_1 = \frac{2}{\sqrt{5}}\left[\ln (4t)^{\frac{1}{2}} - \ln\left(\sqrt{5}+\sqrt{5+4t}\right)\right]+c_1 =$$

$$= \frac{2}{\sqrt{5}}\left[\frac{1}{2}\ln(4\,\mathrm{e}^x) - \ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)\right]+c_1 = \frac{2}{\sqrt{5}}\left[\frac{1}{2}\ln 4 + \frac{1}{2}\ln \mathrm{e}^x - \ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)\right]+c_1 =$$

$$= \frac{\ln 4}{\sqrt{5}} + \frac{x}{\sqrt{5}} - \frac{2}{\sqrt{5}}\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)+c_1 = \frac{x}{\sqrt{5}} - \frac{2}{\sqrt{5}}\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)+c, \qquad \text{for } x\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{5+4\,\mathrm{e}^x}} = \boxed{\begin{array}{c} \mathrm{e}^x=t,\ 4t=u^2,\ u=\frac{\sqrt{5}}{v} \implies \mathrm{e}^x = t = \frac{u^2}{4} = \frac{5}{4v^2},\ v^2 = \frac{5}{4\,\mathrm{e}^x} \\ v=\frac{\sqrt{5}}{2\sqrt{\mathrm{e}^x}},\quad x = \ln\frac{5}{4v^2} = \ln 5 - \ln 4 - 2\ln v \implies \mathrm{d}x = \frac{-2\,\mathrm{d}v}{v} \end{array}} = \int \frac{1}{\sqrt{5+4\frac{5}{4v^2}}}\,\frac{-2\,\mathrm{d}v}{v} =$$

$$= -\frac{2}{\sqrt{5}}\int \frac{\mathrm{d}v}{\sqrt{v^2+1}} = -\frac{2}{\sqrt{5}}\ln\left(v+\sqrt{v^2+1}\right)+c_1 = -\frac{2}{\sqrt{5}}\ln\left(\frac{\sqrt{5}}{2\sqrt{\mathrm{e}^x}}+\sqrt{\frac{5}{4\,\mathrm{e}^x}+1}\right)+c_1 =$$

$$= -\frac{2}{\sqrt{5}}\ln\frac{\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}}{2\sqrt{\mathrm{e}^x}}+c_1 = -\frac{2}{\sqrt{5}}\ln\frac{\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}}{2\,(\mathrm{e}^x)^{\frac{1}{2}}}+c_1 = -\frac{2}{\sqrt{5}}\left[\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right) - \ln 2 - \frac{1}{2}\ln \mathrm{e}^x\right]+c_1 =$$

$$= \frac{2\ln 2}{\sqrt{5}} + \frac{x}{\sqrt{5}} - \frac{2}{\sqrt{5}}\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)+c_1 = \frac{x}{\sqrt{5}} - \frac{2}{\sqrt{5}}\ln\left(\sqrt{5}+\sqrt{5+4\,\mathrm{e}^x}\right)+c, \qquad \text{for } x\in R.$$

$$\int f\left(x,\ \sqrt[m]{\frac{ax+b}{cx+d}}\right) \mathrm{d}x \quad \textbf{for } ad-bc\neq 0$$

$$\boxed{\text{Substitution} \quad t=\sqrt[m]{\frac{ax+b}{cx+d}}} \implies t^m=\frac{ax+b}{cx+d} \implies x=\frac{dt^m-b}{a-ct^m} \implies \mathrm{d}x=\frac{mt^{m-1}(ad-bc)}{(a-ct^m)^2}\,\mathrm{d}t$$

$$\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt[3]{x+1}} = \int \frac{\mathrm{d}x}{\left(\sqrt[6]{x+1}\right)^3+\left(\sqrt[6]{x+1}\right)^2} = \boxed{\begin{array}{c}\sqrt[6]{x+1}=t\\ x+1=t^6\\ \mathrm{d}x=6t^5\,\mathrm{d}t\end{array}} = \int \frac{6t^5\,\mathrm{d}t}{t^3+t^2} = 6\int \frac{t^3\,\mathrm{d}t}{t+1} =$$

$$= 6\int \frac{t^3+t^2-t^2-t+t+1-1}{t+1}\,\mathrm{d}t = 6\int \left[\frac{t^3+t^2}{t+1}-\frac{t^2+t}{t+1}+\frac{t+1}{t+1}-\frac{1}{t+1}\right]\mathrm{d}t = 6\int\left[t^2-t+1-\frac{1}{t+1}\right]\mathrm{d}t =$$

$$= 6\left[\frac{t^3}{3}-\frac{t^2}{2}+t-\ln|t+1|\right]+c = 2\left(\sqrt[6]{x+1}\right)^3-3\left(\sqrt[6]{x+1}\right)^2+6\sqrt[6]{x+1}-6\ln\left|1+\sqrt[6]{x+1}\right|+c =$$

$$= 2\sqrt{x+1}-3\sqrt[3]{x+1}+6\sqrt[6]{x+1}-6\ln\left|1+\sqrt[6]{x+1}\right|+c, \qquad \text{for } \ x>-1.$$

$$\int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\,\mathrm{d}x = \int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\cdot\frac{\sqrt{1-\sqrt{x}}}{\sqrt{1-\sqrt{x}}}\,\mathrm{d}x = \int \frac{1-\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{1-x}}-\int\frac{\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x =$$

$$\boxed{\begin{array}{c}1-x=u\\ \mathrm{d}x=-\mathrm{d}u\end{array}} \quad \text{resp.} \quad \boxed{\frac{x}{1-x}=t^2 \implies x=t^2-xt^2 \implies x=\frac{t^2}{1+t^2} \implies \mathrm{d}x=\frac{2t(1+t^2)-t^2\cdot 2t}{(1+t^2)^2}\,\mathrm{d}t=\frac{2t\,\mathrm{d}t}{(1+t^2)^2}}$$

$$= \int (1-x)^{-\frac{1}{2}}\,\mathrm{d}x-\int\sqrt{\frac{x}{1-x}}\,\mathrm{d}x = -\int u^{-\frac{1}{2}}\,\mathrm{d}u-\int\frac{t\cdot 2t\,\mathrm{d}t}{(1+t^2)^2} = -\frac{u^{\frac{1}{2}}}{\frac{1}{2}}-2\int\frac{t^2\,\mathrm{d}t}{(1+t^2)^2} = -2\sqrt{u}-2\int\frac{1+t^2-1}{(1+t^2)^2}\,\mathrm{d}t =$$

$$= -2\sqrt{1-x}-2\int\left[\frac{1}{1+t^2}-\frac{1}{(1+t^2)^2}\right]\mathrm{d}t = -2\sqrt{1-x}-2\left[\operatorname{arctg} t-\frac{1}{2}\operatorname{arctg} t-\frac{1}{2}\frac{t}{t^2+1}\right]+c =$$

$$\boxed{t^2+1=\frac{x}{1-x}+1=\frac{x+1-x}{1-x}=\frac{1}{1-x} \implies \frac{t}{t^2+1}=(1-x)\sqrt{\frac{x}{1-x}}=\sqrt{(1-x)x}=\sqrt{x-x^2}}$$

$$= -2\sqrt{1-x}-\operatorname{arctg} t+\frac{t}{t^2+1}+c = -2\sqrt{1-x}-\operatorname{arctg}\sqrt{\frac{x}{1-x}}+\sqrt{x-x^2}+c, \qquad \text{for } \ x\in\langle 0\,;\,1).$$

$\int f(x, \sqrt{ax^2+bx+c})\,\mathrm{d}x$ **for** $a\neq 0$

$$\boxed{\text{First Euler substitution: } \sqrt{ax^2+bx+c} = t \pm x\sqrt{a} \quad \text{for } a>0} \implies t = \sqrt{ax^2+bx+c} \mp x\sqrt{a}$$

$$\implies ax^2+bx+c = t^2 \pm 2tx\sqrt{a} + ax^2 \implies x = \frac{t^2-c}{b \mp 2t\sqrt{a}} \implies \mathrm{d}x = 2\frac{\mp t^2\sqrt{a} + tb \mp c\sqrt{a}}{(b \mp 2t\sqrt{a})^2}\,\mathrm{d}t$$

$$\boxed{\text{Second Euler substitution: } \sqrt{ax^2+bx+c} = xt \pm \sqrt{c} \quad \text{for } c>0} \implies t = \frac{\sqrt{ax^2+bx+c} \mp \sqrt{c}}{x}$$

$$\implies ax^2+bx+c = x^2t^2 \pm 2tx\sqrt{c} + c \overset{x\neq 0}{\implies} x = \frac{\pm 2t\sqrt{c} - b}{a-t^2} \implies \mathrm{d}x = 2\frac{\pm t^2\sqrt{c} - tb \pm a\sqrt{c}}{(a-t^2)^2}\,\mathrm{d}t$$

$$\boxed{\text{Third Euler substitution: } t = \sqrt{a\frac{x-u}{x-v}}, \quad \text{if } u,v \text{ are real roots of the } ax^2+bx+c=0} \implies x = \frac{vt^2-au}{t^2-a} \implies \mathrm{d}x = \frac{2ta(u-v)}{(t^2-a)^2}\,\mathrm{d}t$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \boxed{\begin{array}{l} \text{1-st:} \quad \sqrt{x^2+a^2} = t-x \implies x^2+a^2 = t^2-2tx+x^2 \implies 2tx = t^2-a^2 \implies x = \dfrac{t^2-a^2}{2t} \\ \sqrt{x^2+a^2} = t - \dfrac{t^2-a^2}{2t} = \dfrac{t^2+a^2}{2t}, \qquad \mathrm{d}x = \dfrac{2t\cdot 2t - 2(t^2-a^2)}{4t^2}\,\mathrm{d}t = \dfrac{2t^2+2a^2}{4t^2}\,\mathrm{d}t = \dfrac{t^2+a^2}{2t^2}\,\mathrm{d}t \\ \sqrt{a^2+x^2} > \sqrt{x^2} \geq x \implies \sqrt{a^2+x^2} - x > 0 \implies \ln\left|x+\sqrt{a^2+x^2}\right| = \ln\left(x+\sqrt{a^2+x^2}\right) \end{array}} =$$

$$= \int \frac{2t}{t^2+a^2}\frac{t^2+a^2}{2t^2}\,\mathrm{d}t = \int\frac{\mathrm{d}t}{t} = \ln|t| + c = \ln\left|x+\sqrt{x^2+a^2}\right| + c = \ln\left(x+\sqrt{x^2+a^2}\right) + c, \qquad \text{for } a\in R,\ x\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \boxed{\begin{array}{l} \text{2-nd:} \quad \sqrt{a^2-x^2} = xt+a \implies a^2-x^2 = x^2t^2+2txa+a^2 \overset{x\neq 0}{\implies} x = \dfrac{-2ta}{t^2+1},\ t = \dfrac{\sqrt{a^2-x^2}-a}{x} \\ \sqrt{a^2-x^2} = \dfrac{-2t^2a}{t^2+1} + a = \dfrac{a(1-t^2)}{t^2+1}, \qquad \mathrm{d}x = \dfrac{-2a(t^2+1)+2ta\cdot 2t}{(t^2+1)^2}\,\mathrm{d}t = \dfrac{2a(t^2-1)}{(t^2+1)^2}\,\mathrm{d}t \end{array}} =$$

$$= \int \frac{t^2+1}{a(1-t^2)}\frac{2a(t^2-1)}{(t^2+1)^2}\,\mathrm{d}t = -2\int\frac{\mathrm{d}t}{t^2+1} = -2\operatorname{arctg} t + c = -2\operatorname{arctg}\frac{\sqrt{a^2-x^2}-a}{x} + c,$$

for $a\in R,\ a\neq 0,\ x\in(-|a|\,;\,|a|)-\{0\}$.

$$\int \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \boxed{\begin{array}{l} \text{3-rd:} \quad t = \sqrt{\dfrac{x}{1-x}} > 0 \implies t^2 = \dfrac{x}{1-x} \implies t^2 - t^2x = x \implies x = \dfrac{t^2}{t^2+1} \\ \mathrm{d}x = \dfrac{2t(t^2+1)-t^2\cdot 2t}{(t^2+1)^2}\,\mathrm{d}t = \dfrac{2t\,\mathrm{d}t}{(t^2+1)^2},\ x(1-x) = \dfrac{t^2}{t^2+1}\left(1-\dfrac{t^2}{t^2+1}\right) = \dfrac{t^2}{(t^2+1)^2} \end{array}} = \int\frac{t^2+1}{t}\frac{2t\,\mathrm{d}t}{(t^2+1)^2} =$$

$$= \int\frac{2\,\mathrm{d}t}{t^2+1} = 2\operatorname{arctg} t + c = 2\operatorname{arctg}\sqrt{\frac{x}{1-x}} + c, \qquad \text{for } x(1-x) = x-x^2 > 0 \Rightarrow x > x^2 \Rightarrow x\in(0\,;\,1).$$

$\int f(\sin x, \cos x)\,\mathrm{d}x$

$$\boxed{\text{Universal trig substitution:} \quad t = \operatorname{tg}\frac{x}{2}} \implies x = 2\operatorname{arctg} t \implies \mathrm{d}x = \frac{2\,\mathrm{d}t}{t^2+1}$$

$$\implies \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \cdot \frac{\cos^{-2}\frac{x}{2}}{\cos^{-2}\frac{x}{2}} = \frac{2\operatorname{tg}\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\implies \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \cdot \frac{\cos^{-2}\frac{x}{2}}{\cos^{-2}\frac{x}{2}} = \frac{1-\operatorname{tg}^2\frac{x}{2}}{\operatorname{tg}^2\frac{x}{2}+1} = \frac{1-t^2}{t^2+1}$$

$$\int \frac{\mathrm{d}x}{\cos x} = \boxed{t = \operatorname{tg}\frac{x}{2},\ \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2},\ \cos x = \frac{1-t^2}{1+t^2}} = \int \frac{1+t^2}{1-t^2}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{2\,\mathrm{d}t}{1-t^2} = -\int \frac{2\,\mathrm{d}t}{t^2-1} = -\frac{2}{2}\ln\left|\frac{t-1}{t+1}\right| + c =$$

$$= \ln\left|\frac{t+1}{t-1}\right| + c = \ln\left|\frac{\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}\frac{x}{2}-1}\right| + c, \qquad \text{for } x \in R - \left\{(2k+1)\frac{\pi}{2};\, k \in Z\right\}.$$

$$\int \frac{\mathrm{d}x}{1+\cos x} = \boxed{t = \operatorname{tg}\frac{x}{2},\ \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2},\ 1+\cos x = 1+\frac{1-t^2}{1+t^2} = \frac{2}{1+t^2}} = \int \frac{1+t^2}{2}\frac{2\,\mathrm{d}t}{1+t^2} = \int \mathrm{d}t = t + c = \operatorname{tg}\frac{x}{2} + c,$$

$$\text{for } x \in R - \{\pi + 2k\pi;\, k \in Z\}.$$

$$\int \frac{\mathrm{d}x}{\sin x} = \boxed{t = \operatorname{tg}\frac{x}{2},\ \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2},\ \sin x = \frac{2t}{1+t^2}} = \int \frac{1+t^2}{2t}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{\mathrm{d}t}{t} = \ln|t| + c = \ln\left|\operatorname{tg}\frac{x}{2}\right| + c, \qquad \text{for } x \in R - \{k\pi;\, k \in Z\}.$$

$$\int \frac{\mathrm{d}x}{1+\sin x} = \boxed{t = \operatorname{tg}\frac{x}{2},\ \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2},\ 1+\sin x = 1+\frac{2t}{1+t^2} = \frac{1+2t+t^2}{1+t^2}} = \int \frac{1+t^2}{(1+t)^2}\frac{2\,\mathrm{d}t}{1+t^2} = \int \frac{2\,\mathrm{d}t}{(1+t)^2} =$$

$$= \boxed{\begin{matrix} 1+t = u \\ \mathrm{d}t = \mathrm{d}u \end{matrix}} = 2\int \frac{\mathrm{d}u}{u^2} = 2\int u^{-2}\,\mathrm{d}u = 2\frac{u^{-1}}{-1} + c = c - \frac{2}{t+1} = c - \frac{2}{\operatorname{tg}\frac{x}{2}+1}, \qquad \text{for } x \in R - \left\{\frac{\pi}{2} + k\pi;\, k \in Z\right\}.$$

Integration by parts (per partes)

Theorem.
Let f, g be function differentiable on an interval I. Then fg' is integrable on I if and only if $f'g$ is integrable on I, and

$$\forall x \in I\colon \quad \int f'(x)g(x)\,\mathrm{d}x = f(x)g(x) - \int f(x)g'(x)\,\mathrm{d}x.$$

$$\int \operatorname{arctg} x\,\mathrm{d}x = \boxed{\begin{array}{ll} u' = 1 & \Rightarrow\ u = x \\ v = \operatorname{arctg} x & \Rightarrow\ v' = \dfrac{1}{1+x^2} \end{array}} = x\operatorname{arctg} x - \int \frac{x}{1+x^2}\,\mathrm{d}x = x\operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,\mathrm{d}x =$$

$$= x\operatorname{arctg} x - \frac{1}{2}\ln\left|1+x^2\right| + c = x\operatorname{arctg} x - \frac{1}{2}\ln\left(1+x^2\right) + c = x\operatorname{arctg} x - \ln\sqrt{1+x^2} + c, \qquad \text{for } x \in R.$$

$$\int x^2 \ln x\,\mathrm{d}x = \boxed{\begin{array}{ll} u = \ln x & \Rightarrow\ u' = \dfrac{1}{x} \\ v' = x^2 & \Rightarrow\ v = \dfrac{x^3}{3} \end{array}} = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,\mathrm{d}x = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,\mathrm{d}x = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c, \qquad \text{for } x > 0.$$

$$\int \frac{\ln x}{\sqrt{x}}\,\mathrm{d}x = \boxed{\begin{array}{ll} u = \ln x & \Rightarrow\ u' = \dfrac{1}{x} \\ v' = x^{-\frac{1}{2}} & \Rightarrow\ v = 2x^{\frac{1}{2}} \end{array}} = 2\sqrt{x}\ln x - 2\int \frac{x^{\frac{1}{2}}}{x}\,\mathrm{d}x = 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,\mathrm{d}x =$$

$$= 2\sqrt{x}\ln x - 2\cdot 2x^{\frac{1}{2}} + c = 2\sqrt{x}\ln x - 4\sqrt{x} + c, \qquad \text{for } x > 0.$$

$$\int x\ln^2 x\,\mathrm{d}x = \boxed{\begin{array}{ll} u = \ln^2 x & \Rightarrow\ u' = \dfrac{2\ln x}{x} \\ v' = x & \Rightarrow\ v = \dfrac{x^2}{2} \end{array}} = \frac{x^2}{2}\ln^2 x - \int \frac{x^2}{2}\frac{2\ln x}{x}\,\mathrm{d}x = \frac{x^2}{2}\ln^2 x - \int x\ln x\,\mathrm{d}x = \boxed{\begin{array}{ll} u = \ln x & \Rightarrow\ u' = \dfrac{1}{x} \\ v' = x & \Rightarrow\ v = \dfrac{x^2}{2} \end{array}} =$$

$$= \frac{x^2}{2}\ln^2 x - \left[\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,\mathrm{d}x\right] = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \int \frac{x\,\mathrm{d}x}{2} = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + c, \qquad \text{for } x > 0.$$

$$\int \sin^n x \,\mathrm{d}x = \int \sin x \sin^{n-1} x \,\mathrm{d}x = \boxed{\begin{array}{ll} u = \sin^{n-1} x & \Rightarrow \; u' = (n-1)\sin^{n-2} x \cos x \\ v' = \sin x & \Rightarrow \; v = -\cos x \end{array}} =$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x \,\mathrm{d}x = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x (1-\sin^2 x)\,\mathrm{d}x =$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \,\mathrm{d}x - (n-1)\int \sin^n x \,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} \int \sin^n x \,\mathrm{d}x + (n-1)\int \sin^n x \,\mathrm{d}x = -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \,\mathrm{d}x$$

$$\Longrightarrow \int \sin^n x \,\mathrm{d}x = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x \,\mathrm{d}x, \qquad \text{for } \; n = 3, 4, 5, \ldots, \; x \in R.$$

$$\int \cos^n x \,\mathrm{d}x = \int \cos x \cos^{n-1} x \,\mathrm{d}x = \boxed{\begin{array}{ll} u = \cos^{n-1} x & \Rightarrow \; u' = -(n-1)\cos^{n-2} x \sin x \\ v' = \cos x & \Rightarrow \; v = \sin x \end{array}} =$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x \,\mathrm{d}x = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x (1-\cos^2 x)\,\mathrm{d}x =$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \,\mathrm{d}x - (n-1)\int \cos^n x \,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} \int \cos^n x \,\mathrm{d}x + (n-1)\int \cos^n x \,\mathrm{d}x = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \,\mathrm{d}x$$

$$\Longrightarrow \int \cos^n x \,\mathrm{d}x = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x \,\mathrm{d}x, \qquad \text{for } \; n = 3, 4, 5, \ldots, \; x \in R.$$

Partial fractions

$$\int \frac{dx}{(x-a)^n} = \boxed{\begin{matrix} x-a=t \\ dx=du \end{matrix}} = \int \frac{dt}{t^n} = \begin{cases} \int \frac{dt}{t} = \ln|t| + c = \ln|x-a| + c, & \text{for } n=1, \\ \int t^{-n}\,dt = \frac{t^{1-n}}{1-n} + c = \frac{(x-a)^{1-n}}{1-n} + c, & \text{for } n = 2,3,4,\ldots, \end{cases} \quad \text{for } a \in R,\ x \in R - \{a\}.$$

$$\int \frac{dx}{x+1} = \ln|x+1| + c, \qquad \text{for } x \in R - \{-1\}.$$

$$\int \frac{1}{(x-1)^n}\,dx = \boxed{\begin{matrix} x-1=t \\ dx=dt \end{matrix}} = \int t^{-n}\,dt = \frac{t^{1-n}}{1-n} + c = \frac{(x-1)^{1-n}}{1-n} + c, \qquad \text{for } n = 2,3,4,\ldots,\ x \in R - \{1\}.$$

$$\int \frac{dx}{x^2+ax+b} = \boxed{\begin{matrix} x + \frac{a}{2} = t,\ \ \alpha^2 = b - \frac{a^2}{4} > 0 \\ dx = dt,\ \ \alpha = \sqrt{b - a^2/4} \end{matrix}} = \int \frac{dt}{t^2+\alpha^2} = \frac{1}{\alpha}\operatorname{arctg}\frac{t}{\alpha} + c = \frac{1}{\sqrt{b-\frac{a^2}{4}}}\operatorname{arctg}\frac{x+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}} + c.$$

$$\text{for } a, b, A \in R,\ a^2 - 4b < 0,\ x \in R.$$

$$\int \frac{dx}{x^2+4x+25} = \boxed{\begin{matrix} x^2+4x+25 = (x+2)^2+21 \\ x+2=t,\ \ dx=dt \end{matrix}} = \frac{1}{\sqrt{25-\frac{4^2}{4}}}\operatorname{arctg}\frac{x+\frac{4}{2}}{\sqrt{25-\frac{4^2}{4}}} + c = \frac{1}{\sqrt{21}}\operatorname{arctg}\frac{x+2}{\sqrt{21}} + c, \qquad \text{for } x \in R.$$

$$\int \frac{dx}{x^2+b} = \frac{1}{\sqrt{b-\frac{0^2}{4}}}\operatorname{arctg}\frac{x+\frac{0}{2}}{\sqrt{b-\frac{0^2}{4}}} + c = \frac{1}{\sqrt{b}}\operatorname{arctg}\frac{x}{\sqrt{b}} + c, \qquad \text{for } b > 0,\ x \in R.$$

$$\int \frac{dx}{x^2+\beta^2} = \frac{1}{|\beta|}\operatorname{arctg}\frac{x}{|\beta|} + c = \boxed{\frac{1}{-\beta}\operatorname{arctg}\frac{x}{-\beta} = -\frac{1}{\beta}\left[-\operatorname{arctg}\frac{x}{\beta}\right] = \frac{1}{\beta}\operatorname{arctg}\frac{x}{\beta}} = \frac{1}{\beta}\operatorname{arctg}\frac{x}{\beta} + c, \qquad \text{for } \beta \in R,\ x \in R.$$

$$\int \frac{dx}{x^2+1} = \operatorname{arctg} x + c, \qquad \text{for } x \in R.$$

$$\int \frac{x+A}{(x^2+ax+b)^n}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x+a+2A-a}{(x^2+ax+b)^n}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x+a}{(x^2+ax+b)^n}\,\mathrm{d}x + \frac{2A-a}{2}\int \frac{\mathrm{d}x}{(x^2+ax+b)^n} =$$

$$= \begin{cases} \frac{1}{2}\ln\left|x^2+ax+b\right| + \frac{2A-a}{2\sqrt{b-\frac{a^2}{4}}}\operatorname{arctg}\frac{x+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}} + c, & \text{for } n=1,\\ \frac{1}{2(1-n)(x^2+ax+b)^{n-1}} - \frac{2A-a}{4\left(b-\frac{a^2}{4}\right)(1-n)}\frac{x+\frac{a}{2}}{(x^2+ax+b)^{n-1}} + \frac{(3-2n)(2A-a)}{4\left(b-\frac{a^2}{4}\right)(1-n)}\int\frac{\mathrm{d}x}{(x^2+ax+b)^{n-1}}, & \text{for } n=2,3,4,\dots, \end{cases}$$

for $a,b,A\in R,\ a^2-4b<0$ (i.e. $x^2+ax+b=0$ has no real roots), $x\in R$.

$$\int \frac{2x+a}{(x^2+ax+b)^n}\,\mathrm{d}x = \boxed{\begin{array}{c} x^2+ax+b=u\\ (2x+a)\,\mathrm{d}x=\mathrm{d}u \end{array}} = \begin{cases} \int\frac{\mathrm{d}u}{u} = \ln|u|+c = \ln\left|x^2+ax+b\right|+c, & \text{for } n=1,\\ \int\frac{\mathrm{d}u}{u^n} = \int u^{-n}\,\mathrm{d}u = \frac{u^{1-n}}{1-n}+c = \frac{1}{(1-n)(x^2+ax+b)^{n-1}}+c, & \text{for } n=2,3,4,\dots. \end{cases}$$

for $a,b\in R,\ x\in R,\ x\neq\frac{-a\pm\sqrt{a^2-4b}}{2}$.

$$\int\frac{\mathrm{d}x}{(x^2+ax+b)^n} = \boxed{\begin{array}{c} x^2+ax+b=\left(x+\frac{a}{2}\right)^2+b-\frac{a^2}{4},\ \ x+\frac{a}{2}=t\\ \alpha^2=b-\frac{a^2}{4}>0,\ \ \alpha=\sqrt{b-a^2/4},\ \ \mathrm{d}x=\mathrm{d}t \end{array}} = \int\frac{\mathrm{d}t}{(t^2+\alpha^2)^n} = \frac{1}{\alpha^2}\int\frac{\alpha^2}{(t^2+\alpha^2)^n}\,\mathrm{d}t =$$

$$= \frac{1}{\alpha^2}\int\frac{t^2+\alpha^2-t^2}{(t^2+\alpha^2)^n}\,\mathrm{d}t = \frac{1}{\alpha^2}\int\frac{t^2+\alpha^2}{(t^2+\alpha^2)^n}\,\mathrm{d}t - \frac{1}{\alpha^2}\int\frac{t^2}{(t^2+\alpha^2)^n}\,\mathrm{d}t = \frac{1}{\alpha^2}\int\frac{\mathrm{d}t}{(t^2+\alpha^2)^{n-1}} - \frac{1}{\alpha^2}\int\frac{t\cdot t}{(t^2+\alpha^2)^n}\,\mathrm{d}t =$$

$$\boxed{u=t,\ \ u'=1 \implies v'=\frac{t}{(t^2+\alpha^2)^n}=t(t^2+\alpha^2)^{-n},\ \ v=\frac{(t^2+\alpha^2)^{1-n}}{2(1-n)}=\frac{1}{2(1-n)}\frac{1}{(t^2+\alpha^2)^{n-1}}}$$

$$= \frac{1}{\alpha^2}\int\frac{\mathrm{d}t}{(t^2+\alpha^2)^{n-1}} - \frac{1}{\alpha^2}\left[\frac{1}{2(1-n)}\frac{t}{(t^2+\alpha^2)^{n-1}} - \frac{1}{2(1-n)}\int\frac{\mathrm{d}t}{(t^2+\alpha^2)^{n-1}}\right] =$$

$$= \frac{1}{\alpha^2}\left[1+\frac{1}{2(1-n)}\right]\int\frac{\mathrm{d}t}{(t^2+\alpha^2)^{n-1}} - \frac{1}{2\alpha^2(1-n)}\frac{t}{(t^2+\alpha^2)^{n-1}} =$$

$$= \frac{3-2n}{2\alpha^2(1-n)}\int\frac{\mathrm{d}t}{(t^2+\alpha^2)^{n-1}} - \frac{1}{2\alpha^2(1-n)}\frac{t}{(t^2+\alpha^2)^{n-1}} =$$

$$= \frac{3-2n}{2\left(b-\frac{a^2}{4}\right)(1-n)}\int\frac{\mathrm{d}x}{(x^2+ax+b)^{n-1}} - \frac{1}{2\left(b-\frac{a^2}{4}\right)(1-n)}\frac{x+\frac{a}{2}}{(x^2+ax+b)^{n-1}},$$

for $a,b\in R,\ a^2-4b<0,\ n=2,3,4,\dots,\ x\in R$.

$$\int \frac{dx}{(x^2+ax+b)^2} = \frac{1}{2\left(b-\frac{a^2}{4}\right)}\int \frac{dx}{x^2+ax+b} + \frac{1}{2\left(b-\frac{a^2}{4}\right)}\frac{x+\frac{a}{2}}{x^2+ax+b} = \frac{1}{2\left(b-\frac{a^2}{4}\right)^{\frac{3}{2}}}\operatorname{arctg}\frac{x+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}} + \frac{1}{2\left(b-\frac{a^2}{4}\right)}\frac{x+\frac{a}{2}}{x^2+ax+b}+c,$$

for $a, b \in R,\ a^2-4b<0,\ x \in R.$

$$\int \frac{dx}{(x^2+4x+25)^2} = \boxed{\frac{a}{2}=\frac{4}{2}=2,\ b-\frac{a^2}{4}=25-\frac{4^2}{4}=25-4=21} = \frac{1}{2\cdot 21\cdot\sqrt{21}}\operatorname{arctg}\frac{x+2}{\sqrt{21}} + \frac{1}{2\cdot 21}\frac{x+2}{x^2+4x+25}+c,$$ for $x \in R.$

$$\int \frac{dx}{(x^2+b)^2} = \frac{1}{2b}\int \frac{dx}{x^2+b} + \frac{1}{2b}\frac{x}{x^2+b} = \frac{1}{2b\sqrt{b}}\operatorname{arctg}\frac{x}{\sqrt{b}} + \frac{1}{2b}\frac{x}{x^2+b}+c,$$ for $b>0,\ x \in R.$

$$\int \frac{dx}{(x^2+\beta^2)^2} = \frac{1}{2\beta^3}\operatorname{arctg}\frac{x}{\beta} + \frac{1}{2\beta^2}\frac{x}{x^2+\beta^2}+c,$$ for $\beta \in R,\ x \in R.$

$$\int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\operatorname{arctg} x + \frac{1}{2}\frac{x}{x^2+1}+c,$$ for $x \in R.$

$$\int \frac{dx}{(x^2+ax+b)^3} = \frac{3}{4\left(b-\frac{a^2}{4}\right)}\int \frac{dx}{(x^2+ax+b)^2} + \frac{1}{4\left(b-\frac{a^2}{4}\right)}\frac{x+\frac{a}{2}}{(x^2+ax+b)^2}.$$

$$= \frac{3}{4\left(b-\frac{a^2}{4}\right)}\left[\frac{1}{2\left(b-\frac{a^2}{4}\right)^{\frac{3}{2}}}\operatorname{arctg}\frac{x+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}} + \frac{1}{2\left(b-\frac{a^2}{4}\right)}\frac{x+\frac{a}{2}}{x^2+ax+b}\right] + \frac{1}{4\left(b-\frac{a^2}{4}\right)}\frac{x+\frac{a}{2}}{(x^2+ax+b)^2}+c=$$

$$= \frac{3}{8\left(b-\frac{a^2}{4}\right)^{\frac{5}{2}}}\operatorname{arctg}\frac{x+\frac{a}{2}}{\sqrt{b-\frac{a^2}{4}}} + \frac{3}{8\left(b-\frac{a^2}{4}\right)^2}\frac{x+\frac{a}{2}}{x^2+ax+b} + \frac{1}{4\left(b-\frac{a^2}{4}\right)}\frac{x+\frac{a}{2}}{(x^2+ax+b)^2}+c,$$

for $a, b \in R,\ a^2-4b<0,\ x \in R.$

$$\int \frac{dx}{(x^2+4x+25)^3} = \frac{3}{8\cdot 21^2\cdot\sqrt{21}}\operatorname{arctg}\frac{x+2}{\sqrt{21}} + \frac{3}{8\cdot 21^2}\frac{x+2}{x^2+4x+25} + \frac{1}{4\cdot 21}\frac{x+2}{(x^2+4x+25)^2}+c,$$ for $x \in R.$

$$\int \frac{dx}{(x^2+b)^3} = \frac{3}{8b^2\sqrt{b}}\operatorname{arctg}\frac{x}{\sqrt{b}} + \frac{3}{8b^2}\frac{x}{x^2+b} + \frac{1}{4b}\frac{x}{(x^2+b)^2}+c,$$ for $b>0,\ x \in R.$

$$\int \frac{dx}{(x^2+\beta^2)^3} = \frac{3}{8\beta^5}\operatorname{arctg}\frac{x}{\beta} + \frac{3}{8\beta^4}\frac{x}{x^2+\beta^2} + \frac{1}{4\beta^2}\frac{x}{(x^2+\beta^2)^2}+c,$$ for $\beta \in R,\ x \in R.$

$$\int \frac{dx}{(x^2+1)^3} = \frac{3}{8}\operatorname{arctg} x + \frac{3}{8}\frac{x}{x^2+1} + \frac{1}{4}\frac{x}{(x^2+1)^2}+c,$$ for $x \in R.$

$$\int \frac{\mathrm{d}x}{(x^2+ax+b)^n} = \frac{3-2n}{2\left(b-\frac{a^2}{4}\right)(1-n)} \int \frac{\mathrm{d}x}{(x^2+ax+b)^{n-1}} - \frac{1}{2\left(b-\frac{a^2}{4}\right)(1-n)} \frac{x+\frac{a}{2}}{(x^2+ax+b)^{n-1}} =$$

$$= \frac{3-2n}{2\left(b-\frac{a^2}{4}\right)(1-n)} \left[\frac{3-2(n-1)}{2\left(b-\frac{a^2}{4}\right)(1-(n-1))} \int \frac{\mathrm{d}x}{(x^2+ax+b)^{(n-1)-1}} - \frac{1}{2\left(b-\frac{a^2}{4}\right)(1-(n-1))} \frac{x+\frac{a}{2}}{(x^2+ax+b)^{(n-1)-1}} \right] -$$

$$- \frac{1}{2\left(b-\frac{a^2}{4}\right)(1-n)} \frac{x+\frac{a}{2}}{(x^2+ax+b)^{n-1}} =$$

$$= \frac{(3-2n)(5-2n)}{2^2\left(b-\frac{a^2}{4}\right)^2(1-n)(2-n)} \int \frac{\mathrm{d}x}{(x^2+ax+b)^{n-2}} - \frac{3-2n}{2^2\left(b-\frac{a^2}{4}\right)^2(1-n)(2-n)} \frac{x+\frac{a}{2}}{(x^2+ax+b)^{n-2}} -$$

$$- \frac{1}{2\left(b-\frac{a^2}{4}\right)(1-n)} \frac{x+\frac{a}{2}}{(x^2+ax+b)^{n-1}} = \cdots,$$

for $a, b \in R,\ a^2-4b<0,\ n=3,4,5,\ldots,\ x \in R.$

$$\int \frac{\mathrm{d}x}{(x^2+b)^n} = \frac{3-2n}{2b(1-n)} \int \frac{\mathrm{d}x}{(x^2+b)^{n-1}} - \frac{1}{2b(1-n)} \frac{x}{(x^2+b)^{n-1}} =$$

$$= \frac{3-2n}{2b(1-n)} \left[\frac{5-2n}{2b(2-n)} \int \frac{\mathrm{d}x}{(x^2+b)^{n-2}} - \frac{1}{2b(2-n)} \frac{x}{(x^2+b)^{n-2}} \right] - \frac{1}{2b(1-n)} \frac{x}{(x^2+b)^{n-1}} =$$

$$= \frac{(3-2n)(5-2n)}{2^2b^2(1-n)(2-n)} \int \frac{\mathrm{d}x}{(x^2+b)^{n-2}} - \frac{3-2n}{2^2b^2(1-n)(2-n)} \frac{x}{(x^2+b)^{n-2}} - \frac{1}{2b(1-n)} \frac{x}{(x^2+b)^{n-1}} = \cdots,$$

for $b>0,\ n=3,4,5,\ldots,\ x \in R.$

$$\int \frac{\mathrm{d}x}{(x^2+\beta^2)^n} = \frac{3-2n}{2\beta^2(1-n)} \int \frac{\mathrm{d}x}{(x^2+\beta^2)^{n-1}} - \frac{1}{2\beta^2(1-n)} \frac{x}{(x^2+\beta^2)^{n-1}},$$ for $\beta \in R,\ n=2,3,4,\ldots,\ x \in R.$

$$\int \frac{\mathrm{d}x}{(x^2+1)^n} = \frac{3-2n}{2(1-n)} \int \frac{\mathrm{d}x}{(x^2+1)^{n-1}} - \frac{1}{2(1-n)} \frac{x}{(x^2+1)^{n-1}},$$ for $n=2,3,4,\ldots,\ x \in R.$

$$\int \frac{x\,\mathrm{d}x}{(x^2+1)^n} = \boxed{\begin{array}{c} x^2+1=t \\ 2x\,\mathrm{d}x=\mathrm{d}t \end{array}} = \int \frac{t^{-n}\,\mathrm{d}t}{2} = \begin{cases} \frac{1}{2}|t|+c = \frac{1}{2}\ln|x^2+1|+c = \frac{1}{2}\ln(x^2+1)+c = \ln\sqrt{x^2+1}+c, & \text{for } n=1, \\ \frac{1}{2}\frac{t^{-n+1}}{-n+1}+c = \frac{1}{2(1-n)t^{n-1}}+c = \frac{1}{2(1-n)(x^2+1)^{n-1}}+c, & \text{for } n=2,3,4,\ldots, \end{cases}$$ for $x \in R.$

Fourier Series

$$\int_a^a f(x)\,\mathrm{d}x = 0, \quad \int_a^b f(x) = -\int_b^a f(x), \quad \text{for all } a,b\in R,\, f(x)\in R_{\langle a\,;\,b\rangle} \ \text{[i.e. } f(x) \text{ is Riemann integrable on } \langle a\,;\,b\rangle\text{]}$$

Theorem.

The **Fourier series** of a function $f(x)\in R_{\langle a\,;\,a+2\pi\rangle}$, $a\in R$ is given by

$$f(x) \approx A_0\frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[A_n\frac{\cos nx}{\sqrt{\pi}} + B_n\frac{\sin nx}{\sqrt{\pi}}\right], \quad \text{for } x\in\langle a\,;\,a+2\pi\rangle,$$

where $A_0 = \int_a^{a+2\pi}\frac{f(x)}{\sqrt{2\pi}}\,\mathrm{d}x, \quad A_n = \int_a^{a+2\pi} f(x)\frac{\cos nx}{\sqrt{\pi}}\,\mathrm{d}x, \quad B_n = \int_a^{a+2\pi} f(x)\frac{\sin nx}{\sqrt{\pi}}\,\mathrm{d}x, \quad n\in N,$

resp. $f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx], \quad \text{for } x\in\langle a\,;\,a+2\pi\rangle,$

where $a_0 = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\,\mathrm{d}x, \quad a_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\cos nx\,\mathrm{d}x, \quad b_n = \frac{1}{\pi}\int_a^{a+2\pi} f(x)\sin nx\,\mathrm{d}x, \quad n\in N.$

A Fourier series converges to the function
$$\tilde{f}(x_0) = \begin{cases} \frac{1}{2}\left[\lim\limits_{x\to x_0^-} f(x) + \lim\limits_{x\to x_0^+} f(x)\right], & x_0\in(a\,;\,a+2\pi)\,, \\ f(x_0-2k\pi), & x_0\in(a+2k\pi\,;\,a+2\pi+2k\pi)\,, \\ \frac{1}{2}\left[\lim\limits_{x\to a^+} f(x) + \lim\limits_{x\to a+2\pi^-} f(x)\right], & x_0 = a+2k\pi,\ k\in Z. \end{cases}$$

(Function $\tilde{f}(x)$ is a periodic function with period 2π and equals to the original function at the interval $\langle a\,;\,a+2\pi\rangle$ at points of continuity or to the average of the two limits at points of discontinuity.)

$$f(x) \approx \tilde{f}(x) = A_0\frac{1}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[A_n\frac{\cos nx}{\sqrt{\pi}} + B_n\frac{\sin nx}{\sqrt{\pi}}\right] =$$

$$= \frac{1}{\sqrt{2\pi}}\int_a^{a+2\pi}\frac{f(x)}{\sqrt{2\pi}}\,\mathrm{d}x + \sum_{n=1}^{\infty}\left[\frac{\cos nx}{\sqrt{\pi}}\int_a^{a+2\pi} f(x)\frac{\cos nx}{\sqrt{\pi}}\,\mathrm{d}x + \frac{\sin nx}{\sqrt{\pi}}\int_a^{a+2\pi} f(x)\frac{\sin nx}{\sqrt{\pi}}\,\mathrm{d}x\right] =$$

$$= \frac{1}{2\pi}\int_a^{a+2\pi} f(x)\,\mathrm{d}x + \sum_{n=1}^{\infty}\left[\frac{\cos nx}{\pi}\int_a^{a+2\pi} f(x)\cos nx\,\mathrm{d}x + \frac{\sin nx}{\pi}\int_a^{a+2\pi} f(x)\sin nx\,\mathrm{d}x\right] = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n\cos nx + b_n\sin nx],$$

for $x\in\langle a\,;\,a+2\pi\rangle$.

Theorem.

The **Fourier series** of a function $f(x)\in R_{\langle a\,;\,a+2l\rangle}$, $a\in R$, $l>0$ is given by

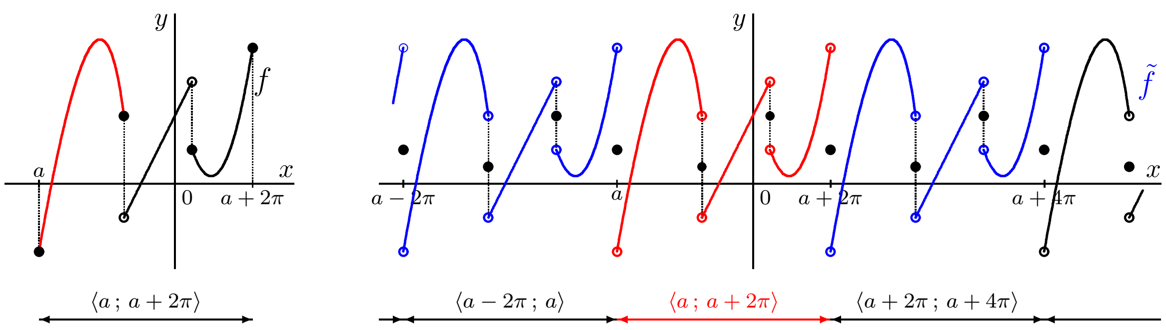
$$f(x)\approx\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\right],\quad \text{for } x\in\langle a\,;\,a+2l\rangle,$$

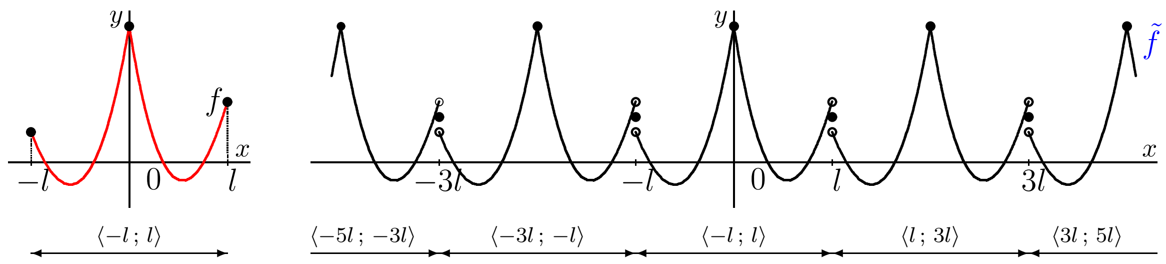
where $a_0=\frac{1}{l}\int_a^{a+2l}f(x)\,\mathrm{d}x$, $a_n=\frac{1}{l}\int_a^{a+2l}f(x)\cos\frac{n\pi x}{l}\,\mathrm{d}x$, $b_n=\frac{1}{l}\int_a^{a+2l}f(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$, $n\in N$.

$$\left[\langle a\,;\,a+2l\rangle\longmapsto\langle a\,;\,a+2\pi\rangle\quad\Longrightarrow\quad x\longmapsto\frac{x}{2l}2\pi=\frac{\pi x}{l}\quad\Longrightarrow\quad nx\longmapsto\frac{n\pi x}{l},\ n\in N\right]$$

A Fourier series converges to the function $\tilde f(x_0)=\begin{cases}\frac{1}{2}\left[\lim\limits_{x\to x_0^-}f(x)+\lim\limits_{x\to x_0^+}f(x)\right], & x_0\in(a\,;\,a+2l)\,,\\ f(x_0-2kl), & x_0\in(a+2l\,;\,a+2l+2kl)\,,\\ \frac{1}{2}\left[\lim\limits_{x\to a^+}f(x)+\lim\limits_{x\to a+2l^-}f(x)\right], & x_0=a+2kl,\ k\in Z.\end{cases}$

(Function $\tilde f(x)$ is a periodic function with period $2l$ and equals to the original function at the interval $\langle a\,;\,a+2l\rangle$ at points of continuity or to the average of the two limits at points of discontinuity.)





Theorem.

(i) If $f(x)$ is an odd function, then $a_n = 0$, $n = 0, 1, 2, \ldots$ and the Fourier series collapses to **Fourier Sine Series**

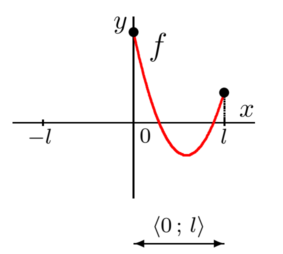
$$f(x) \approx \sum_{n=1}^{\infty} b_n \sin nx, \quad \text{for } x \in \langle -\pi \,;\, \pi \rangle, \quad \text{where} \quad b_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, \mathrm{d}x, \; n \in N.$$

$$\text{resp.} \quad f(x) \approx \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad \text{for } x \in \langle -l \,;\, l \rangle, \; l > 0, \quad \text{where} \quad b_n = \frac{2}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, \mathrm{d}x, \; n \in N.$$

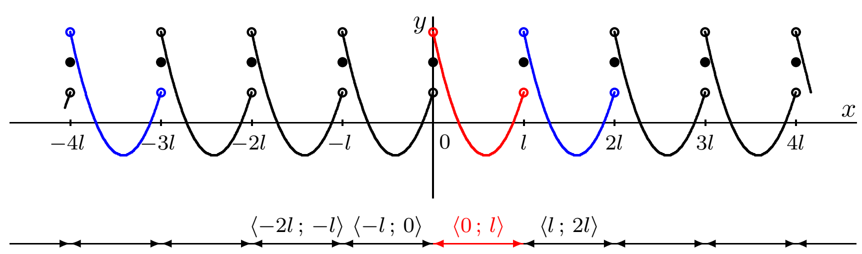
(ii) If $f(x)$ is an even function, then $b_n = 0$, $n \in N$ and the Fourier series collapses to **Fourier Cosine Series**

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \quad \text{for } x \in \langle -\pi \,;\, \pi \rangle, \quad \text{where} \quad a_0 = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \, \mathrm{d}x, \; a_n = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, \mathrm{d}x, \; n \in N.$$

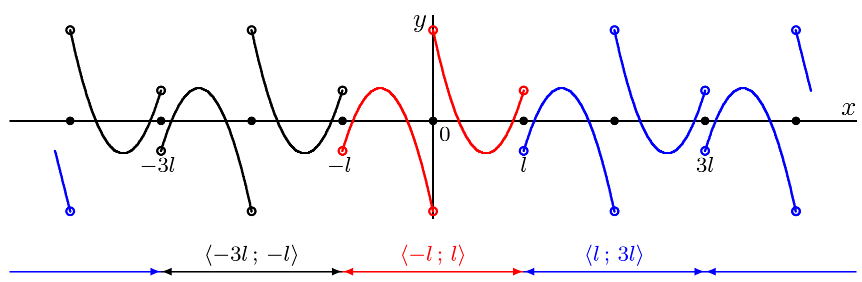
$$\text{resp.} \quad f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad \text{for } x \in \langle -l \,;\, l \rangle, \; l > 0, \quad \text{where} \quad a_0 = \frac{2}{l} \int_{-l}^{l} f(x) \, \mathrm{d}x, \; a_n = \frac{2}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, \mathrm{d}x, \; n \in N.$$



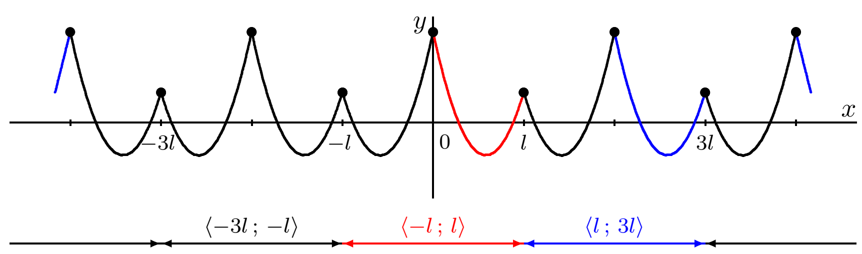
Function $f \in R_{\langle 0\,;\,l \rangle}$



Fourier Series



Fourier Sine Series



Fourier Cosine Series

Theorem.

$$\int_a^{a+2\pi} f(x)\,\mathrm{d}x = \int_0^{2\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \qquad \text{for a periodic function } f(x)\in R_{\langle a\,;\,a+2\pi\rangle} \text{ with period } 2\pi,\ a>0.$$

$$\int_a^{a+2\pi} f(x)\,\mathrm{d}x = \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_{2\pi}^{a+2\pi} f(x)\,\mathrm{d}x = \boxed{\begin{matrix} x=t-2\pi, & \mathrm{d}x=\mathrm{d}t \\ 2\pi\mapsto 0, & a+2\pi\mapsto a\end{matrix}} = \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_0^{a} f(t-2\pi)\,\mathrm{d}t =$$

$$= \boxed{f(t-2\pi)=f(t)} = \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_0^{a} f(t)\,\mathrm{d}t = \int_a^{2\pi} f(x)\,\mathrm{d}x + \int_0^{a} f(x)\,\mathrm{d}x = \int_0^{2\pi} f(x)\,\mathrm{d}x$$

$$\int_0^{2\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \qquad \text{for a periodic function } f(x)\in R_{\langle 0\,;\,2\pi\rangle} \text{ with period } 2\pi.$$

$$\int_a^{b} f(x)\,\mathrm{d}x = \boxed{\begin{matrix} x=-t, & \mathrm{d}x=-\,\mathrm{d}t \\ a\mapsto -a, & b\mapsto -b\end{matrix}} = -\int_{-a}^{-b} f(-t)\,\mathrm{d}t = \int_{-b}^{-a} f(-t)\,\mathrm{d}t = \int_{-b}^{-a} f(-x)\,\mathrm{d}x, \qquad \text{for } f(x)\in R_{\langle a\,;\,b\rangle}.$$

$$\int_a^{b} f(x)\,\mathrm{d}x = \int_{-b}^{-a} f(-x)\,\mathrm{d}x = \begin{cases} \displaystyle\int_{-b}^{-a} f(x)\,\mathrm{d}x, & \text{for a even function } f(x)\in R_{\langle a\,;\,b\rangle}, \\ \displaystyle -\int_{-b}^{-a} f(x)\,\mathrm{d}x, & \text{for a odd function } f(x)\in R_{\langle a\,;\,b\rangle}. \end{cases}$$

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_0^{a} f(x)\,\mathrm{d}x = \begin{cases} \displaystyle\int_0^{a} f(x)\,\mathrm{d}x + \int_0^{a} f(x)\,\mathrm{d}x = 2\int_0^{a} f(x)\,\mathrm{d}x, & \text{for a even function } f(x)\in R_{\langle -a\,;\,a\rangle},\ a>0, \\ \displaystyle -\int_0^{a} f(x)\,\mathrm{d}x + \int_0^{a} f(x)\,\mathrm{d}x = 0, & \text{for a odd function } f(x)\in R_{\langle -a\,;\,a\rangle},\ a>0. \end{cases}$$

$$\sin 2k\pi = 0, \quad \sin k\pi = 0, \quad \sin(2k+1)\frac{\pi}{2} = (-1)^k, \quad \sin(2k-1)\frac{\pi}{2} = (-1)^{k-1} = (-1)^{k+1},$$
$$\cos 2k\pi = 1, \quad \cos k\pi = (-1)^k, \quad \cos(2k+1)\frac{\pi}{2} = 0, \quad \text{for } \ k \in Z$$

$$\int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \int_{-\pi}^{\pi} \cos nx \cdot \cos mx \, dx = \int_a^{a+2\pi} \cos nx \cdot \cos mx \, dx = 0, \qquad \text{for } \ m, n \in N, \ m \neq n, \ a \in R.$$

$$\int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \boxed{\begin{array}{l} u = \cos nx \quad \Rightarrow \quad u' = -n \sin nx \\ v' = \cos mx \quad \Rightarrow \quad v = \dfrac{\sin mx}{m} \end{array}} = \left[\frac{\cos nx \cdot \sin mx}{m}\right]_0^{2\pi} + \frac{n}{m} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx =$$

$$= \left[\frac{1 \cdot 0}{m} - \frac{1 \cdot 0}{m}\right] + \frac{n}{m} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \frac{n}{m} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \boxed{\begin{array}{l} u = \sin nx \quad \Rightarrow \quad u' = n \cos nx \\ v' = \sin mx \quad \Rightarrow \quad v = -\dfrac{\cos mx}{m} \end{array}} =$$

$$= \frac{n}{m} \left(\left[-\frac{\sin nx \cdot \cos mx}{m}\right]_0^{2\pi} + \frac{n}{m} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx \right) = \frac{n}{m} \left[-\frac{0 \cdot 1}{m} + \frac{0 \cdot 1}{m}\right]_0^{2\pi} + \frac{n^2}{m^2} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx$$

$$\xRightarrow{\text{(i.e.equation)}} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \frac{n^2}{m^2} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx \Longrightarrow \left(1 - \frac{n^2}{m^2}\right) \int_0^{2\pi} \cos nx \cdot \cos mx \, dx = 0$$

$$\xRightarrow{m \neq n} 1 - \frac{n^2}{m^2} = \frac{m^2 - n^2}{m^2} \neq 0 \Longrightarrow \int_0^{2\pi} \cos nx \cdot \cos mx \, dx = 0, \qquad \text{for } \ m, n \in N, \ m \neq n.$$

$$\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \int_{-\pi}^{\pi} \sin nx \cdot \sin mx \, dx = \int_a^{a+2\pi} \sin nx \cdot \sin mx \, dx = 0, \qquad \text{for } \ m, n \in N, \ m \neq n, \ a \in R.$$

$$\int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \boxed{\begin{array}{l} u = \sin nx \quad \Rightarrow \quad u' = n \cos nx \\ v' = \sin mx \quad \Rightarrow \quad v = -\dfrac{\cos mx}{m} \end{array}} = \left[-\frac{\sin nx \cdot \cos mx}{m}\right]_0^{2\pi} + \frac{n}{m} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx =$$

$$= \left[-\frac{0 \cdot 1}{m} + \frac{0 \cdot 1}{m}\right] + \frac{n}{m} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \frac{n}{m} \int_0^{2\pi} \cos nx \cdot \cos mx \, dx = \boxed{\begin{array}{l} u = \cos nx \quad \Rightarrow \quad u' = -n \sin nx \\ v' = \cos mx \quad \Rightarrow \quad v = \dfrac{\sin mx}{m} \end{array}} =$$

$$= \frac{n}{m} \left(\left[\frac{\cos nx \cdot \sin mx}{m}\right]_0^{2\pi} + \frac{n}{m} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx \right) = \frac{n}{m} \left[\frac{1 \cdot 0}{m} - \frac{1 \cdot 0}{m}\right]_0^{2\pi} + \frac{n^2}{m^2} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx$$

$$\xRightarrow{\text{(i.e.equation)}} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \frac{n^2}{m^2} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx \Longrightarrow \left(1 - \frac{n^2}{m^2}\right) \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0$$

$$\xRightarrow{m \neq n} 1 - \frac{n^2}{m^2} = \frac{m^2 - n^2}{m^2} \neq 0 \Longrightarrow \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = 0, \qquad \text{for } \ m, n \in N, \ m \neq n.$$

$$\int_0^{2\pi} \cos nx \cdot \sin mx \, dx = \int_{-\pi}^{\pi} \cos nx \cdot \sin mx \, dx = \int_a^{a+2\pi} \cos nx \cdot \sin mx \, dx = 0, \qquad \text{for } \; m, n \in N, \; a \in R.$$

$$\int_0^{2\pi} \cos nx \cdot \sin mx \, dx = \boxed{\begin{array}{lll} u = \cos nx & \Rightarrow & u' = -n \sin nx \\ v' = \sin mx & \Rightarrow & v = -\dfrac{\cos mx}{m} \end{array}} = \left[-\frac{\cos nx \cdot \cos mx}{m} \right]_0^{2\pi} - \frac{n}{m} \int_0^{2\pi} \sin nx \cdot \cos mx \, dx =$$

$$= \left[-\frac{1 \cdot 1}{m} + \frac{1 \cdot 1}{m} \right] - \frac{n}{m} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = -\frac{n}{m} \int_0^{2\pi} \sin nx \cdot \sin mx \, dx = \boxed{\begin{array}{lll} u = \sin nx & \Rightarrow & u' = n \cos nx \\ v' = \cos mx & \Rightarrow & v = \dfrac{\sin mx}{m} \end{array}} =$$

$$= -\frac{n}{m} \left(\left[\frac{\sin nx \cdot \sin mx}{m} \right]_0^{2\pi} - \frac{n}{m} \int_0^{2\pi} \cos nx \cdot \sin mx \, dx \right) = -\frac{n}{m} \left[\frac{0 \cdot 0}{m} - \frac{0 \cdot 0}{m} \right]_0^{2\pi} + \frac{n^2}{m^2} \int_0^{2\pi} \cos nx \cdot \sin mx \, dx$$

$$\xRightarrow{\text{(i.e.equation)}} \int_0^{2\pi} \cos nx \cdot \sin mx \, dx = \frac{n^2}{m^2} \int_0^{2\pi} \cos nx \cdot \sin mx \, dx \Longrightarrow \left(1 - \frac{n^2}{m^2} \right) \int_0^{2\pi} \cos nx \cdot \sin mx \, dx = 0$$

$$\xRightarrow{m \neq n} 1 - \frac{n^2}{m^2} = \frac{m^2 - n^2}{m^2} \neq 0 \Longrightarrow \int_0^{2\pi} \cos nx \cdot \sin mx \, dx = 0, \qquad \text{for } \; m, n \in N, \; m \neq n.$$

$$\int_0^{2\pi} \cos nx \cdot \sin nx \, dx = \frac{1}{2} \int_0^{2\pi} \sin 2nx \, dx = \frac{1}{2} \left[-\frac{\cos 2nx}{2n} \right]_0^{2\pi} = \frac{1}{2} \left[-\frac{\cos 4n\pi}{2n} + \frac{\cos 0}{2n} \right] = \frac{1}{2} \left[-\frac{1}{2n} + \frac{1}{2n} \right] = 0, \qquad \text{for } \; m = n.$$

$$\int_0^{2\pi} \cos nx \, dx = \int_{-\pi}^{\pi} \cos nx \, dx = \int_a^{a+2\pi} \cos nx \, dx = \left[\frac{\sin nx}{n} \right]_a^{a+2\pi} = \frac{\sin(na+2n\pi) - \sin na}{n} = \frac{\sin na - \sin na}{n} = 0, \quad \text{for } n \in N, \; a \in R.$$

$$\int_0^{2\pi} \sin nx \, dx = \int_{-\pi}^{\pi} \sin nx \, dx = \int_a^{a+2\pi} \sin nx \, dx = \left[-\frac{\cos nx}{n} \right]_a^{a+2\pi} = \frac{\cos na - \cos(na+2n\pi)}{n} = \frac{\cos na - \cos na}{n} = 0, \quad \text{for } n \in N, \; a \in R.$$

$$\|1\| = \sqrt{\int_0^{2\pi} 1^2\,\mathrm{d}x} = \sqrt{\int_{-\pi}^{\pi} 1^2\,\mathrm{d}x} = \sqrt{\int_a^{a+2\pi} 1^2\,\mathrm{d}x} = \sqrt{\int_a^{a+2\pi} \mathrm{d}x} = \sqrt{a+2\pi-a} = \sqrt{2\pi}, \qquad \text{for} \quad a\in R.$$

$$\|\cos nx\| = \sqrt{\int_0^{2\pi} \cos^2 nx\,\mathrm{d}x} = \sqrt{\int_{-\pi}^{\pi} \cos^2 nx\,\mathrm{d}x} = \sqrt{\int_a^{a+2\pi} \cos^2 nx\,\mathrm{d}x} = \sqrt{\pi}, \qquad \text{for} \quad n\in N,\ a\in R.$$

$$\int_a^{a+2\pi} \cos^2 nx\,\mathrm{d}x = \int_a^{a+2\pi} \frac{1+\cos 2nx}{2}\,\mathrm{d}x = \frac{1}{2}\int_a^{a+2\pi} \mathrm{d}x + \frac{1}{2}\int_a^{a+2\pi} \cos 2nx\,\mathrm{d}x = \frac{a+2\pi-a}{2} + \frac{1}{2}\cdot 0 = \pi.$$

$$\|\sin nx\| = \sqrt{\int_0^{2\pi} \sin^2 nx\,\mathrm{d}x} = \sqrt{\int_{-\pi}^{\pi} \sin^2 nx\,\mathrm{d}x} = \sqrt{\int_a^{a+2\pi} \sin^2 nx\,\mathrm{d}x} = \sqrt{\pi}, \qquad \text{for} \quad n\in N,\ a\in R.$$

$$\int_a^{a+2\pi} \sin^2 nx\,\mathrm{d}x = \int_a^{a+2\pi} \frac{1-\cos 2nx}{2}\,\mathrm{d}x = \frac{1}{2}\int_a^{a+2\pi} \mathrm{d}x - \frac{1}{2}\int_a^{a+2\pi} \cos 2nx\,\mathrm{d}x = \frac{a+2\pi-a}{2} - \frac{1}{2}\cdot 0 = \pi.$$

$$\boxed{\frac{\pi x}{l} = t, \quad x = \frac{lt}{\pi}, \quad \mathrm{d}x = \frac{l}{\pi}\,\mathrm{d}t, \quad x = a \implies t = \frac{\pi a}{l} = b, \quad x = a+2l \implies t = \frac{\pi(a+2l)}{l} = \frac{\pi a}{l} + 2\pi = b+2\pi}$$

$$\int_a^{a+2l} \cos\frac{n\pi x}{l}\,\mathrm{d}x = \frac{l}{\pi}\int_b^{b+2\pi} \cos nt\,\mathrm{d}t = \frac{l}{\pi}\cdot 0 = 0, \qquad \text{for} \quad n\in N,\ a\in R,\ l>0.$$

$$\int_a^{a+2l} \sin\frac{n\pi x}{l}\,\mathrm{d}x = \frac{l}{\pi}\int_b^{b+2\pi} \sin nt\,\mathrm{d}t = \frac{l}{\pi}\cdot 0 = 0, \qquad \text{for} \quad n\in N,\ a\in R,\ l>0.$$

$$\int_0^{2\pi} \cos\frac{n\pi x}{l}\cdot\cos\frac{m\pi x}{l}\,\mathrm{d}x = \frac{l}{\pi}\int_a^{a+2\pi} \cos nt\cdot\cos mt\,\mathrm{d}t = \frac{l\cdot 0}{\pi} = 0, \qquad \text{for} \quad m,n\in N,\ m\neq n,\ a\in R,\ l>0.$$

$$\int_0^{2\pi} \sin\frac{n\pi x}{l}\cdot\sin\frac{m\pi x}{l}\,\mathrm{d}x = \frac{l}{\pi}\int_a^{a+2\pi} \sin nt\cdot\sin mt\,\mathrm{d}t = \frac{l\cdot 0}{\pi} = 0, \qquad \text{for} \quad m,n\in N,\ m\neq n,\ a\in R,\ l>0.$$

$$\int_0^{2\pi} \cos\frac{n\pi x}{l}\cdot\sin\frac{m\pi x}{l}\,\mathrm{d}x = \frac{l}{\pi}\int_a^{a+2\pi} \cos nt\cdot\sin mt\,\mathrm{d}t = \frac{l\cdot 0}{\pi} = 0, \qquad \text{for} \quad m,n\in N,\ a\in R,\ l>0.$$

$$\|1\| = \sqrt{\int_a^{a+2l} 1^2\,\mathrm{d}x} = \sqrt{[x]_a^{a+2l}} = \sqrt{2l}, \qquad \text{for} \quad a\in R,\ l>0.$$

$$\left\|\cos\frac{n\pi x}{l}\right\| = \sqrt{\int_a^{a+2l} \cos^2\frac{n\pi x}{l}\,\mathrm{d}x} = \sqrt{\frac{l}{\pi}\int_b^{b+2\pi} \cos^2 nt\,\mathrm{d}t} = \sqrt{\frac{l}{\pi}\pi} = \sqrt{l}, \qquad \text{for} \quad n\in N,\ a\in R,\ l>0.$$

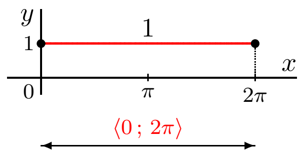
$$\left\|\sin\frac{n\pi x}{l}\right\| = \sqrt{\int_a^{a+2l} \sin^2\frac{n\pi x}{l}\,\mathrm{d}x} = \sqrt{\frac{l}{\pi}\int_b^{b+2\pi} \sin^2 nt\,\mathrm{d}t} = \sqrt{\frac{l}{\pi}\pi} = \sqrt{l}, \qquad \text{for} \quad n\in N,\ a\in R,\ l>0.$$

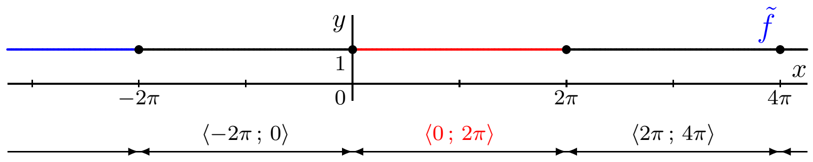
$f(x)=1,\ x\in\langle 0\,;\,2\pi\rangle$

$$\xrightarrow{\text{Fourier series}} f(x)\approx\tilde f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]=\frac{2}{2}+\sum_{n=1}^{\infty}[0\cdot\cos nx+0\cdot\sin nx]=1+\sum_{n=1}^{\infty}0=1,\quad x\in\langle 0\,;\,2\pi\rangle.$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\mathrm{d}x=\frac{2\pi}{\pi}=2,\quad a_n=\frac{1}{\pi}\int_0^{2\pi}\cos nx\,\mathrm{d}x=\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{\sin 2n\pi}{n}+\frac{\sin 0}{n}\right]=\frac{1}{\pi}[0-0]=0,\ n\in N.$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\sin nx\,\mathrm{d}x=\frac{1}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{2\pi}=\frac{1}{\pi}\left[-\frac{\cos 2n\pi}{n}+\frac{\cos 0}{n}\right]=\frac{1}{\pi}\left[-\frac{1}{n}+\frac{1}{n}\right]=0,\ n\in N.$$

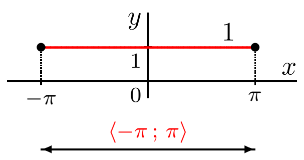


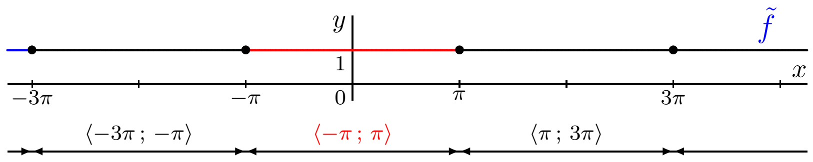


$f(x)=1,\ x\in\langle -\pi\,;\,\pi\rangle$ $\quad$ [$f(x)$ is a even function, $\quad$ i.e. $b_n=0,\ n=1,2,3,\ldots$]

$$\xrightarrow{\text{Fourier Cosine series}} f(x)\approx\tilde f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{2}{2}+\sum_{n=1}^{\infty}0\cdot\cos nx=1+\sum_{n=1}^{\infty}0=1,\quad x\in\langle -\pi\,;\,\pi\rangle.$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}\mathrm{d}x=\frac{2}{\pi}\pi=2,\quad a_n=\frac{2}{\pi}\int_0^{\pi}\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\sin n\pi}{n}-\frac{\sin 0}{n}\right]=\frac{2}{\pi}\left[\frac{0}{n}-\frac{0}{n}\right]=0,\quad n\in N.$$





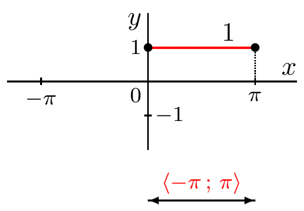
$$f(x)=1,\ x\in\langle 0\,;\,\pi\rangle\ \Longrightarrow\ f_o(x)=\begin{cases}1, & x\in(0\,;\,\pi\rangle\,,\\ 0, & x=0,\\ -1, & x\in\langle -\pi\,;\,0)\end{cases}\quad\text{is a odd function,}\quad\text{i.e. } a_n=0,\ n=0,1,2,\ldots$$

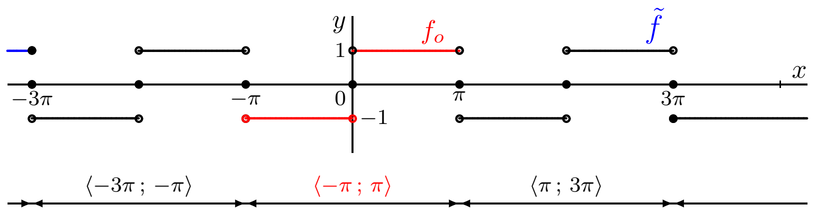
$$\xrightarrow{\text{Fourier Sine series}} f_o(x)\approx\tilde f_o(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\frac{2[(-1)^{n+1}+1]\sin nx}{n\pi}=\sum_{k=1}^{\infty}\frac{4\sin(2k-1)x}{(2k-1)\pi},\quad x\in\langle -\pi\,;\,\pi\rangle.$$

$$\xrightarrow{\text{Fourier Sine series}} f(x)\approx\tilde f_o(x)=\sum_{k=1}^{\infty}\frac{4\sin(2k-1)x}{(2k-1)\pi}=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1},\quad x\in(0\,;\,\pi).$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f_o(x)\sin nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}\sin nx\,\mathrm{d}x=\frac{2}{\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi}=\frac{2}{\pi}\left[-\frac{\cos n\pi}{n}+\frac{\cos 0}{n}\right]=\frac{2}{\pi}\left[-\frac{(-1)^n}{n}+\frac{1}{n}\right]=$$

$$=\frac{2}{\pi}\,\frac{(-1)^{n+1}+1}{n}=\frac{2[(-1)^{n+1}+1]}{n\pi}=\begin{cases}\dfrac{2(1+1)}{n\pi}=\dfrac{4}{n\pi}=\dfrac{4}{(2k-1)\pi}, & \text{for } n=2k-1,\\[2ex] \dfrac{2(-1+1)}{n\pi}=0, & \text{for } n=2k,\ k\in N.\end{cases}$$





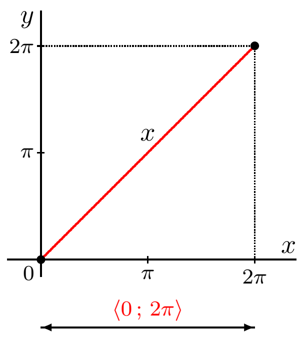
$f(x)=x,\ x\in\langle 0\,;\,2\pi\rangle$

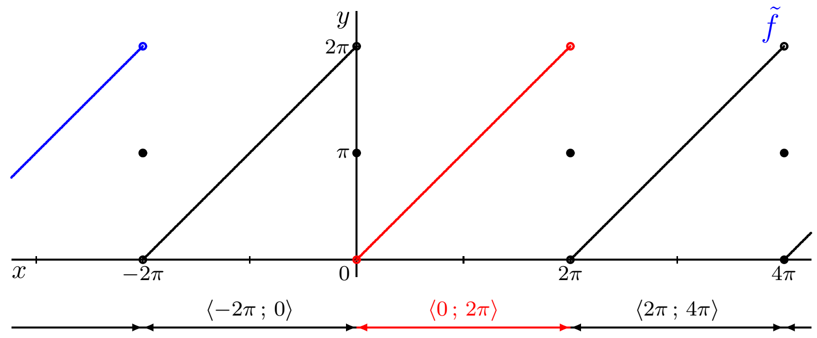
$$\xrightarrow{\text{Fourier series}} f(x)\approx\tilde f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]=\pi+\sum_{n=1}^{\infty}\left[0\cdot\cos nx-\frac{2}{n}\sin nx\right]=\pi-2\sum_{n=1}^{\infty}\frac{\sin nx}{n},\qquad x\in(0\,;\,2\pi).$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x\,\mathrm{d}x=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}\mathrm{d}x=\frac{4\pi^2}{2\pi}=2\pi,$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x\cos nx\,\mathrm{d}x\ \underset{73}{\overset{\text{page}}{=\!=}}\ \frac{1}{\pi}\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{2\pi\cdot 0}{n}+\frac{1}{n^2}-\frac{0\cdot 0}{n}-\frac{1}{n^2}\right]=0,\qquad n\in N,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\,\mathrm{d}x\ \underset{72}{\overset{\text{page}}{=\!=}}\ \frac{1}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{2\pi}=\frac{1}{\pi}\left[-\frac{2\pi\cdot 1}{n}+\frac{0}{n^2}+\frac{0\cdot 1}{n}-\frac{0}{n^2}\right]=-\frac{2}{n},\qquad n\in N.$$





$f(x)=x,\ x\in\langle 0\,;\,\pi\rangle \implies f_e(x)=|x|=\begin{cases}x, & x\in\langle 0\,;\,\pi\rangle\,,\\ -x, & x\in\langle -\pi\,;\,0)\end{cases}$ is a even function, $b_n=0,\ n=1,2,3,\ldots$

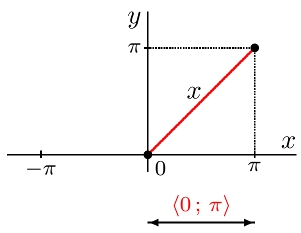
$$\xrightarrow{\text{Fourier Cosine series}} f_e(x)\approx\tilde f_e(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2[(-1)^n-1]\cos nx}{n^2\pi}=\frac{\pi}{2}-\sum_{k=1}^{\infty}\frac{4\cos(2k-1)\pi}{(2k-1)^2\pi},\qquad x\in\langle -\pi\,;\,\pi\rangle.$$

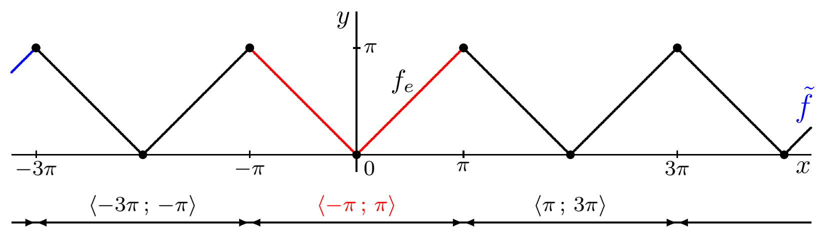
$$\xrightarrow{\text{Fourier Cosine series}} f(x)\approx\tilde f_e(x)=\frac{\pi}{2}-\sum_{k=1}^{\infty}\frac{4\cos(2k-1)\pi}{(2k-1)^2\pi}=\frac{\pi}{2}-4\sum_{k=1}^{\infty}\frac{\cos(2k-1)\pi}{(2k-1)^2\pi},\qquad x\in\langle 0\,;\,\pi\rangle.$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f_e(x)\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}|x|\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}x\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{\pi}\frac{\pi^2}{2}=\pi,$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f_e(x)\cos nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\pi\sin n\pi}{n}+\frac{\cos n\pi}{n^2}-\frac{0\cdot\sin 0}{n}-\frac{\cos 0}{n^2}\right]=$$

$$=\frac{2}{\pi}\left[0+\frac{(-1)^n}{n^2}-0-\frac{1}{n^2}\right]=\frac{2}{\pi}\frac{(-1)^n-1}{n^2}=\begin{cases}\dfrac{2(-1-1)}{n^2\pi}=\dfrac{-4}{n^2\pi}=\dfrac{-4}{(2k-1)^2\pi}, & \text{for } n=2k-1,\\[2mm] \dfrac{2(1-1)}{n^2\pi}=0, & \text{for } n=2k,\ k\in N.\end{cases}$$





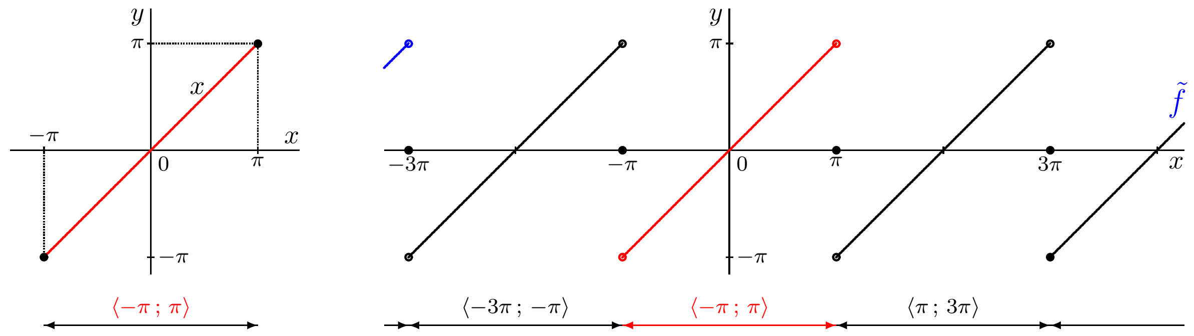
$f(x)=x,\ x\in\langle-\pi\,;\,\pi\rangle$ $\quad[f(x)$ is a odd function, $\quad$ i.e. $a_n=0,\ n=0,1,2,\dots]$

$$\xrightarrow{\text{Fourier Sine series}} f(x)\approx\tilde f(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n},\qquad x\in\langle-\pi\,;\,\pi\rangle.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx\,\mathrm{d}x=\frac{1}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_{-\pi}^{\pi}=\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}+\frac{\sin n\pi}{n^2}+\frac{-\pi\cos[-n\pi]}{n}-\frac{\sin[-n\pi]}{n^2}\right]=$$

$$=\frac{1}{\pi}\left[-\frac{\pi\cos n\pi}{n}+\frac{0}{n^2}-\frac{\pi\cos n\pi}{n}-\frac{0}{n^2}\right]=-\frac{2\cos n\pi}{n}=-\frac{2(-1)^n}{n}=\frac{2(-1)^{n+1}}{n},\qquad n\in N,$$

$$\text{resp.}\quad b_n=\frac{2}{\pi}\int_0^{\pi}x\sin nx\,\mathrm{d}x=\frac{2}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}=\frac{2}{\pi}\left[-\frac{\pi(-1)^n}{n}+\frac{0}{n^2}+\frac{0\cdot 1}{n}-\frac{0}{n^2}\right]=\frac{2(-1)^{n+1}}{n},\qquad n\in N.$$



$$\tilde f(x)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n},\ x\in\langle-\pi\,;\,\pi\rangle,\qquad \tilde f(x)=f(x)=x,\ x\in(-\pi\,;\,\pi),\quad \tilde f(-\pi)=\tilde f(\pi)=\frac{f(-\pi)+f(\pi)}{2}=\frac{-\pi+\pi}{2}=0.$$

$$f\left(\frac{\pi}{2}\right)=\frac{\pi}{2},$$

$$\tilde f\left(\frac{\pi}{2}\right)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}=2\left[\sin\frac{\pi}{2}-\frac12\sin\pi+\frac13\sin\frac{3\pi}{2}-\frac14\sin 2\pi+\frac15\sin\frac{5\pi}{2}-\frac16\sin 3\pi+\cdots\right]=$$

$$=2\left[1-\frac12\cdot 0+\frac13(-1)-\frac14\cdot 0+\frac15\cdot 1-\frac16\cdot 0+\cdots\right]=2\left[1-\frac13+\frac15-\frac17+\frac19+\cdots\right]=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1},$$

$$\text{resp.}\ \tilde f\left(\frac{\pi}{2}\right)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}=2\sum_{k=1}^{\infty}\left[\frac{(-1)^{(2k-1)+1}}{2k-1}\sin\frac{(2k-1)\pi}{2}+\frac{(-1)^{2k+1}}{2k}\sin\frac{2k\pi}{2}\right]=$$

$$=2\sum_{k=1}^{\infty}\left[\frac{(-1)^{2k}}{2k-1}\sin\frac{(2k-1)\pi}{2}+\frac{(-1)^{2k+1}}{2k}\sin k\pi\right]=2\sum_{k=1}^{\infty}\left[\frac{1}{2k-1}\cdot(-1)^{k+1}-\frac{1}{2k}\cdot 0\right]=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}.$$

$$\frac{\pi}{2}=f\left(\frac{\pi}{2}\right)=\tilde f\left(\frac{\pi}{2}\right)=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\quad\Longrightarrow\quad\frac{\pi}{4}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\quad\Longrightarrow\quad\frac{\pi}{4}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=1-\frac13+\frac15-\frac17+\cdots+\frac{(-1)^{n+1}}{2n-1}+\cdots.$$

$f(x)=x^2,\ x\in\langle 0\,;\,2\pi\rangle$

$$\xrightarrow{\text{Fourier series}} f(x)\approx\tilde{f}(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}[a_n\cos nx+b_n\sin nx]=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left[\frac{4\cos nx}{n^2}-\frac{4\pi\sin nx}{n}\right],\quad x\in\langle 0\,;\,2\pi\rangle.$$

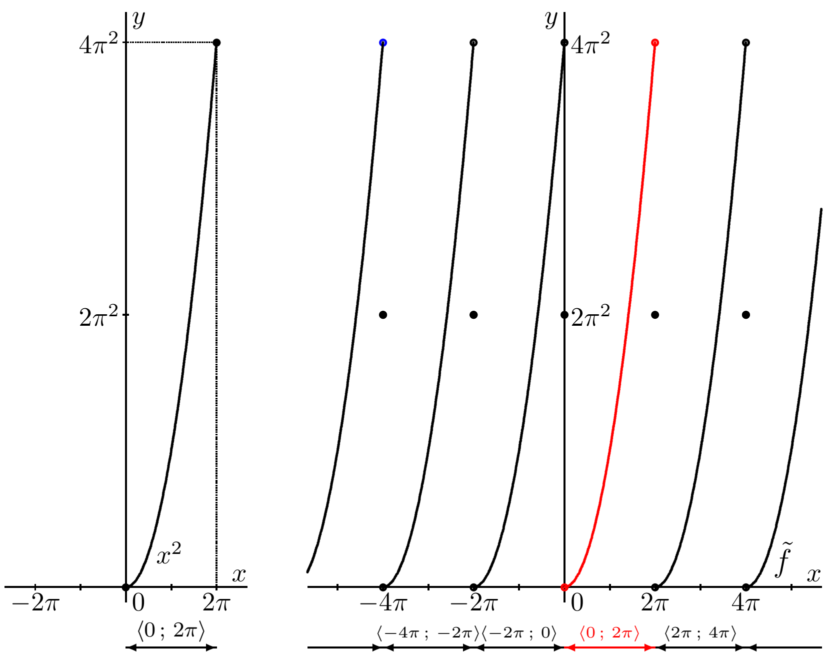
$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2\,\mathrm{d}x=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{(2\pi)^3}{3}-\frac{0}{3}\right]=\frac{8\pi^3}{3\pi}=\frac{8\pi^2}{3},$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,\mathrm{d}x\ \underset{73}{\overset{\text{page}}{=\!=}}\ \frac{1}{\pi}\left[\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right]_0^{2\pi}=\frac{1}{\pi}\left[\frac{4\pi^2\sin 2n\pi}{n}+\frac{4\pi\cos 2n\pi}{n^2}-\frac{2\sin 2n\pi}{n^3}-\right.$$

$$\left.-\frac{0\cdot\sin 0}{n}-\frac{0\cdot\cos 0}{n^2}+\frac{2\sin 0}{n^3}\right]=\frac{1}{\pi}\left[0+\frac{4\pi}{n^2}-0-0-0+0\right]=\frac{4}{n^2},\quad n\in N,$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin nx\,\mathrm{d}x\ \underset{72}{\overset{\text{page}}{=\!=}}\ \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}\right]_0^{2\pi}=\frac{1}{\pi}\left[-\frac{4\pi^2\cos 2n\pi}{n}+\frac{4\pi\sin 2n\pi}{n^2}+\frac{2\cos 2n\pi}{n^3}+\right.$$

$$\left.+\frac{0\cdot\cos 0}{n}-\frac{0\cdot\sin 0}{n^2}-\frac{2\cos 0}{n^3}\right]=\frac{1}{\pi}\left[-\frac{4\pi^2}{n}+0+\frac{2}{n^3}+0-0-\frac{2}{n^3}\right]=-\frac{4\pi}{n},\quad n\in N.$$



$f(x)=x^2,\ x\in\langle 0\,;\,\pi\rangle \implies f_o(x)=x\cdot|x|=\begin{cases} x^2, & x\in(0\,;\,\pi\rangle,\\ -x^2, & x\in\langle -\pi\,;\,0)\end{cases}$ is a odd function, $a_n=0,\ n=0,1,2,\ldots$

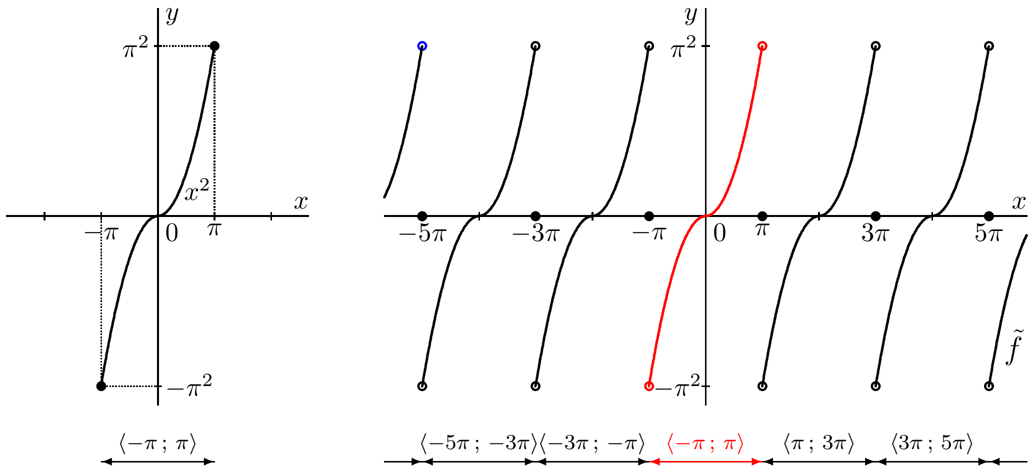
$$\xrightarrow{\text{Fourier Sine series}} f_o(x)\approx\tilde{f}(x)=\sum_{n=1}^{\infty}b_n\sin nx=\sum_{n=1}^{\infty}\left[\frac{2\pi(-1)^{n+1}}{n}+\frac{4[(-1)^n-1]}{n^3\pi}\right]\sin nx,\quad x\in\langle -\pi\,;\,\pi\rangle.$$

$$\xrightarrow{\text{Fourier Sine series}} f(x)\approx\tilde{f}(x)=\sum_{n=1}^{\infty}\left[\frac{2\pi(-1)^{n+1}}{n}+\frac{4[(-1)^n-1]}{n^3\pi}\right]\sin nx=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n-(-1)^n\pi^2n^2-2}{n^3}\sin nx,\quad x\in\langle 0\,;\,\pi\rangle.$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f_o(x)\sin nx\,\mathrm{d}x=\frac{2}{\pi}\int_0^{\pi}x^2\sin nx\,\mathrm{d}x=\frac{2}{\pi}\left[-\frac{x^2\cos nx}{n}+\frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}\right]_0^{\pi}=$$

$$=\frac{2}{\pi}\left[-\frac{\pi^2\cos n\pi}{n}+\frac{2\pi\sin n\pi}{n^2}+\frac{2\cos n\pi}{n^3}+\frac{0\cdot\cos 0}{n}-\frac{0\cdot\sin 0}{n^2}-\frac{2\cos 0}{n^3}\right]=$$

$$=\frac{2}{\pi}\left[-\frac{\pi^2(-1)^n}{n}+0+\frac{2(-1)^n}{n^3}+0-0-\frac{2}{n^3}\right]=\frac{2\pi(-1)^{n+1}}{n}+\frac{4[(-1)^n-1]}{n^3\pi}=\frac{2}{\pi}\frac{2(-1)^n-(-1)^n\pi^2n^2-2}{n^3},\quad n\in N.$$



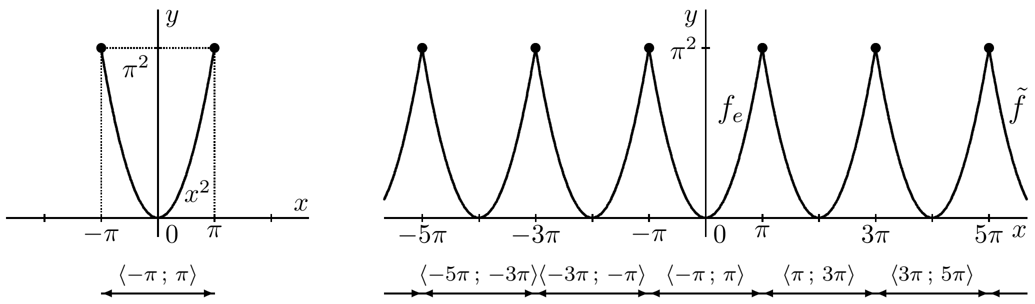
$f(x)=x^2,\ x\in\langle -\pi\,;\,\pi\rangle$ $\quad$ [$f(x)$ is a even function, $\quad b_n=0$, $n=1,2,3,\ldots$]

$$\xRightarrow{\text{Fourier Cosine series}} f(x)\approx\tilde f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n\cos nx}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2},\quad x\in\langle -\pi\,;\,\pi\rangle.$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}x^2\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\pi^3}{3}-\frac{0}{3}\right]=\frac{2\pi^3}{3\pi}=\frac{2\pi^2}{3},$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}x^2\cos nx\,\mathrm{d}x=\frac{2}{\pi}\left[\frac{x^2\sin nx}{n}+\frac{2x\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right]_0^{\pi}=\frac{2}{\pi}\left[\frac{\pi^2\sin n\pi}{n}+\frac{2\pi\cos n\pi}{n^2}-\frac{2\sin n\pi}{n^3}-\right.$$

$$\left.-\frac{0\cdot\sin 0}{n}-\frac{0\cdot\cos 0}{n^2}+\frac{2\sin 0}{n^3}\right]=\frac{2}{\pi}\left[0+\frac{2\pi(-1)^n}{n^2}-0-0-0+0\right]=\frac{4(-1)^n}{n^2},\quad n\in N.$$



$$\tilde f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2},\ x\in\langle -\pi\,;\,\pi\rangle,\qquad \tilde f(x)=f(x)=x^2,\ x\in\langle -\pi\,;\,\pi\rangle.$$

$$f(0)=0,\quad \tilde f(0)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos(0\cdot n)}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cdot 1}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2},$$

$$0=f(0)=\tilde f(0)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\implies\frac{\pi^2}{3}=-4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\implies\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=1-\frac14+\frac19-\frac1{16}+\cdots+\frac{(-1)^{n+1}}{n^2}+\cdots.$$

$$f(\pi)=\pi^2,\quad \tilde f(\pi)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n\cos nx}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2}=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2},$$

$$\pi^2=f(\pi)=\tilde f(\pi)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{1}{n^2}\implies\frac{2\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{1}{n^2}\implies\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac14+\frac19+\frac1{16}+\cdots+\frac{1}{n^2}+\cdots.$$

$$f(x)=x,\quad F(x)\xlongequal[\text{page28}]{\text{Fourier (Sine) series}} 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n},\quad x\in\langle-\pi\,;\,\pi\rangle.$$

$$g(x)=x^2,\quad G(x)\xlongequal[\text{page30}]{\text{Fourier (Cosine) series}} \frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos nx}{n^2},\quad x\in\langle-\pi\,;\,\pi\rangle.$$

$$\forall x\in R:\quad [x^2]'=2x,\quad x^2=\int_0^x 2t\,\mathrm{d}t$$

$$\Longrightarrow\quad \forall x\in(-\pi\,;\,\pi):\ f(x)=\frac{g'(x)}{2},\quad F(x)=\frac{G'(x)}{2},\quad g(x)=2\int_0^x f(t)\,\mathrm{d}t,\quad G(x)=2\int_0^x F(t)\,\mathrm{d}t.$$

$$G(x)=2\int_0^x F(t)\,\mathrm{d}t=2\int_0^x\left[2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nt}{n}\right]\mathrm{d}t=4\sum_{n=1}^{\infty}\int_0^x\frac{(-1)^{n+1}\sin nt}{n}\,\mathrm{d}t=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^x\sin nt\,\mathrm{d}t=$$

$$=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left[\frac{-\cos nt}{n}\right]_0^x=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-\cos nx+1)}{n^2}=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}-4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cos nx}{n^2}$$

$$\xRightarrow{\text{Fourier series}}\quad 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{a_0}{2}\quad\Longrightarrow\quad \frac{a_0}{2}=\frac{1}{2}\frac{2}{\pi}\int_0^{\pi}g(x)\,\mathrm{d}x=\frac{1}{\pi}\int_0^{\pi}x^2\,\mathrm{d}x=\frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}\mathrm{d}x=\frac{1}{\pi}\frac{\pi^3}{3}=\frac{\pi^2}{3}$$

$$\Longrightarrow\quad 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{3}\quad\Longrightarrow\quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}.$$

$$\xRightarrow{\text{Fourier (Cosine) series}}\quad g(x)=x^2,\quad G(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos nx}{n^2},\quad x\in\langle-\pi\,;\,\pi\rangle.$$

$$F(x)=\frac{G'(x)}{2}=\frac{1}{2}\left[\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos nx}{n^2}\right]'=\frac{1}{2}\cdot 4\left[\sum_{n=1}^{\infty}\frac{(-1)^{n}\cos nx}{n^2}\right]'=2\sum_{n=1}^{\infty}\left[\frac{(-1)^{n}\cos nx}{n^2}\right]'=$$

$$=2\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n^2}\,[\cos nx]'=2\sum_{n=1}^{\infty}\frac{(-1)^{n}(-n)\sin nx}{n^2}=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n}$$

$$\xRightarrow{\text{Fourier (Sine) series}}\quad f(x)=x,\quad F(x)=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin nx}{n},\quad x\in\langle-\pi\,;\,\pi\rangle.$$

Integration and Series

$$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots \rightrightarrows \text{ for } x\in R,$$

$$e^{-x} = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!} \rightrightarrows, \qquad e^{(-x^2)} = \sum_{k=0}^{\infty}\frac{(-x^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!} \rightrightarrows, \qquad e^{(x^2)} = \sum_{k=0}^{\infty}\frac{(x^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{x^{2k}}{k!} \rightrightarrows \text{ for } x\in R.$$

$$\int e^x\,dx = \int\left[\sum_{k=0}^{\infty}\frac{x^k}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{x^k}{k!}\,dx = \sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)k!} + c_1 = \sum_{k=0}^{\infty}\frac{x^{k+1}}{(k+1)!} + c_1 = \boxed{\begin{matrix}k+1=n\\ k=0,\ n=1\end{matrix}} = c_1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} = \boxed{1=\frac{x^0}{0!}} =$$

$$= c_1 - 1 + \frac{x^0}{0!} + \sum_{n=1}^{\infty}\frac{x^n}{n!} = \boxed{c_1 - 1 = c} = c + \sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x + c, \qquad \text{for } x\in R.$$

$$\int e^x\,dx = \int\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots\right]dx = \int dx + \int x\,dx + \int\frac{x^2}{2!}\,dx + \int\frac{x^3}{3!}\,dx + \cdots + \int\frac{x^k}{k!}\,dx + \cdots =$$

$$= c_1 + x + \frac{x^2}{2} + \frac{x^3}{3\cdot 2!} + \frac{x^4}{4\cdot 3!} + \cdots + \frac{x^{k+1}}{(k+1)k!} + \cdots = c_1 - 1 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^{k+1}}{(k+1)!} + \cdots = e^x + c, \quad \text{for } x\in R.$$

$$\int e^{-x}\,dx = \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^k}{k!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{k+1}}{(k+1)k!} + c_1 = \boxed{\begin{matrix}k+1=n\\ k=0,\ n=1\end{matrix}} = c_1 + \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n!} =$$

$$= c_1 - \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!} = \boxed{1=\frac{x^0}{0!}} = c_1 + 1 - \left[\frac{x^0}{0!} + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!}\right] = \boxed{c_1 + 1 = c} = c - \sum_{n=0}^{\infty}\frac{(-1)^n x^n}{n!} = -e^{-x} + c, \quad \text{for } x\in R.$$

$$\int e^{-x}\,dx = \int\left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^k x^k}{k!} + \cdots\right]dx = \int dx - \int x\,dx + \int\frac{x^2}{2!}\,dx - \int\frac{x^3}{3!}\,dx + \cdots + \int\frac{(-1)^k x^k}{k!}\,dx + \cdots =$$

$$= c_1 + x - \frac{x^2}{2} + \frac{x^3}{3\cdot 2!} - \frac{x^4}{4\cdot 3!} + \cdots + \frac{(-1)^k x^{k+1}}{(k+1)k!} + \cdots = c_1 + 1 - \left[1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^{k+1} x^{k+1}}{(k+1)!} + \cdots\right] = -e^x + c,$$
for $x\in R$.

$$\int e^{(-x^2)}\,dx = \int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)k!} + c, \qquad \text{for } x\in R.$$

$$\int e^{(-x^2)}\,dx = \int\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^k x^{2k}}{k!} + \cdots\right]dx = \int dx - \int x^2\,dx + \int\frac{x^4}{2!}\,dx - \int\frac{x^6}{3!}\,dx + \cdots + \int\frac{(-1)^k x^{2k}}{k!}\,dx + \cdots =$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)k!} + \cdots + c = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)k!} + c, \qquad \text{for } x\in R.$$

$$\int e^{(x^2)}\,dx = \int\left[\sum_{k=0}^{\infty}\frac{kx^{2k}}{k!}\right]dx = \sum_{k=0}^{\infty}\int\frac{x^{2k}}{k!}\,dx = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)k!} + c, \qquad \text{for } x\in R.$$

$$\int e^{(x^2)}\,dx = \int\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots + \frac{x^{2k}}{k!} + \cdots\right]dx = \int dx + \int x^2\,dx + \int\frac{x^4}{2!}\,dx + \int\frac{x^6}{3!}\,dx + \cdots + \int\frac{x^{2k}}{k!}\,dx + \cdots =$$

$$= x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} + \cdots + \frac{x^{2k+1}}{(2k+1)k!} + \cdots + c = \sum_{k=0}^{\infty}\frac{x^{2k+1}}{(2k+1)k!} + c, \qquad \text{for } x\in R.$$

$$\sin x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)!}+\cdots\rightrightarrows \text{ for } x\in R, \qquad \frac{\sin x}{x}=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!}\rightrightarrows \text{ for } x\in R,$$

$$\cos x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^k x^{2k}}{(2k)!}+\cdots\rightrightarrows \text{ for } x\in R, \qquad \frac{\cos x}{x}=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k-1}}{(2k)!}\rightrightarrows \text{ for } x\in R-\{0\}.$$

$$\int\sin x\,\mathrm{d}x=\int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}\right]\mathrm{d}x=\sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k+1}}{(2k+1)!}\,\mathrm{d}x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+2}}{(2k+2)(2k+1)!}+c_1=\boxed{\begin{matrix}k+1=n, k=0\\ 2k+2=2n, n=1\end{matrix}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{(2n)!}+c_1=$$

$$=\boxed{1=\frac{(-1)^0x^{2\cdot 0}}{(2\cdot 0)!}}=c_1+1-\frac{(-1)^0x^{2\cdot 0}}{(2\cdot 0)!}-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=\boxed{c_1+1=c}=c-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}=-\cos x+c,\ \text{ for } x\in R.$$

$$\int\sin x\,\mathrm{d}x=\int\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)!}+\ldots\right]\mathrm{d}x=\int x\,\mathrm{d}x-\int\frac{x^3}{3!}\,\mathrm{d}x+\int\frac{x^5}{5!}\,\mathrm{d}x-\cdots+\int\frac{(-1)^k x^{2k+1}}{(2k+1)!}\,\mathrm{d}x+\cdots=$$

$$=\frac{x^2}{2}-\frac{x^4}{4\cdot 3!}+\frac{x^6}{6\cdot 5!}-\cdots+\frac{(-1)^k x^{2k+2}}{(2k+2)(2k+1)!}+\cdots+c_1=-1+\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\cdots+\frac{(-1)^{k-1}x^{2k}}{(2k)!}+\frac{(-1)^k x^{2k+2}}{(2k+2)!}+\cdots+c_1+1=$$

$$=-\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\cdots+\frac{(-1)^k x^{2k}}{(2k)!}+\frac{(-1)^{k+1}x^{2k+2}}{(2k+2)!}+\cdots\right]+c=-\cos x+c, \qquad \text{for } x\in R.$$

$$\int\frac{\sin x}{x}\,\mathrm{d}x=\int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k+1)!}\right]\mathrm{d}x=\sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k}}{(2k+1)!}\,\mathrm{d}x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}+c, \qquad \text{for } x\in R-\{0\}.$$

$$\int\frac{\sin x}{x}\,\mathrm{d}x=\int\left[1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots+\frac{(-1)^k x^{2k}}{(2k+1)!}+\cdots\right]\mathrm{d}x=\int\mathrm{d}x-\int\frac{x^2}{3!}\,\mathrm{d}x+\int\frac{x^4}{5!}\,\mathrm{d}x-\int\frac{x^6}{7!}\,\mathrm{d}x+\cdots+\int\frac{(-1)^k x^{2k}}{(2k+1)!}\,\mathrm{d}x+\cdots=$$

$$=x-\frac{x^3}{3\cdot 3!}+\frac{x^5}{5\cdot 5!}-\frac{x^7}{7\cdot 7!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)(2k+1)!}+\cdots+c, \qquad \text{for } x\in R-\{0\}.$$

$$\int\cos x\,\mathrm{d}x=\int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}\right]\mathrm{d}x=\sum_{k=0}^{\infty}\int\frac{(-1)^k x^{2k}}{(2k)!}\,\mathrm{d}x=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)(2k)!}+c=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}+c=\sin x+c, \qquad \text{for } x\in R.$$

$$\int\cos x\,\mathrm{d}x=\int\left[1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^k x^{2k}}{(2k)!}+\cdots\right]\mathrm{d}x=\int\mathrm{d}x-\int\frac{x^2}{2!}\,\mathrm{d}x+\int\frac{x^4}{4!}\,\mathrm{d}x-\int\frac{x^6}{6!}\,\mathrm{d}x+\cdots+\int\frac{(-1)^k x^{2k}}{(2k)!}\,\mathrm{d}x+\cdots=$$

$$=x-\frac{x^3}{3\cdot 2!}+\frac{x^5}{5\cdot 4!}-\frac{x^7}{7\cdot 6!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)(2k)!}+\cdots+c=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^k x^{2k+1}}{(2k+1)!}+\cdots+c=\sin x+c,$$

for $x\in R$.

$$\int\frac{\cos x}{x}\,\mathrm{d}x=\int\left[\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k-1}}{(2k)!}\right]\mathrm{d}x=\int\left[\frac{1}{x}+\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k-1}}{(2k)!}\right]\mathrm{d}x=\int\frac{\mathrm{d}x}{x}+\sum_{k=1}^{\infty}\int\frac{(-1)^k x^{2k-1}}{(2k)!}\,\mathrm{d}x=\ln|x|+\sum_{k=1}^{\infty}\frac{(-1)^k x^{2k}}{2k(2k)!},$$

for $x\in R-\{0\}$.

$$\int\frac{\cos x}{x}\,\mathrm{d}x=\int\left[\frac{1}{x}-\frac{x}{2!}+\frac{x^3}{4!}-\frac{x^5}{6!}+\cdots+\frac{(-1)^k x^{2k-1}}{(2k)!}+\cdots\right]\mathrm{d}x=\int\frac{\mathrm{d}x}{x}-\int\frac{x}{2!}\,\mathrm{d}x+\int\frac{x^3}{4!}\,\mathrm{d}x-\int\frac{x^5}{6!}\,\mathrm{d}x+\cdots+\int\frac{(-1)^k x^{2k}}{2k(2k)!}\,\mathrm{d}x+\cdots=$$

$$=\ln|x|-\frac{x^2}{2\cdot 2!}+\frac{x^4}{4\cdot 4!}-\frac{x^6}{6\cdot 6!}+\cdots+\frac{(-1)^k x^{2k}}{2k(2k)!}+\cdots+c, \qquad \text{for } x\in R-\{0\}.$$

Examples

$$\int \left[\sqrt{x^3} - \frac{1}{\sqrt{x}}\right] \mathrm{d}x = \int \left[x^{\frac{3}{2}} - x^{-\frac{1}{2}}\right] \mathrm{d}x = \frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + c = \frac{2\sqrt{x^5}}{5} - 2\sqrt{x} + c, \qquad \text{for } x > 0.$$

$$\int x(x-a)(x-b)\,\mathrm{d}x = \int \left[x^3 - (a+b)x^2 + abx\right] \mathrm{d}x = \frac{1}{4}x^4 - \frac{a+b}{3}x^3 + \frac{ab}{2}x^2 + c, \qquad \text{for } a, b \in R,\ x \in R.$$

$$\int \mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\int a\,\mathrm{e}^{ax}\,\mathrm{d}x = \boxed{\begin{array}{c} ax = t \\ a\,\mathrm{d}x = \mathrm{d}t \end{array}} = \frac{1}{a}\int \mathrm{e}^{t}\,\mathrm{d}t = \frac{\mathrm{e}^t}{a} + c = \frac{\mathrm{e}^{ax}}{a} + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R.$$

$$\int \sinh^2 x\,\mathrm{d}x = \int \frac{(\mathrm{e}^x - \mathrm{e}^{-x})^2}{4}\,\mathrm{d}x = \frac{1}{4}\int \left[\mathrm{e}^{2x} - 2\,\mathrm{e}^x\,\mathrm{e}^{-x} + \mathrm{e}^{-2x}\right] \mathrm{d}x = \frac{1}{4}\left[\frac{\mathrm{e}^{2x}}{2} - 2x + \frac{\mathrm{e}^{-2x}}{-2}\right] + c = \frac{\mathrm{e}^{2x}}{8} - \frac{\mathrm{e}^{-2x}}{8} - \frac{x}{2} + c, \qquad \text{for } x \in R.$$

$$\int \cosh^2 x\,\mathrm{d}x = \int \frac{(\mathrm{e}^x + \mathrm{e}^{-x})^2}{4}\,\mathrm{d}x = \frac{1}{4}\int \left[\mathrm{e}^{2x} + 2\,\mathrm{e}^x\,\mathrm{e}^{-x} + \mathrm{e}^{-2x}\right] \mathrm{d}x = \frac{1}{4}\left[\frac{\mathrm{e}^{2x}}{2} + 2x + \frac{\mathrm{e}^{-2x}}{-2}\right] + c = \frac{\mathrm{e}^{2x}}{8} - \frac{\mathrm{e}^{-2x}}{8} + \frac{x}{2} + c, \qquad \text{for } x \in R.$$

$$\int [\operatorname{tg} x + \operatorname{cotg} x]\,\mathrm{d}x = \int \left[\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right] \mathrm{d}x = \int \left[\frac{-(\cos x)'}{\cos x} + \frac{(\sin x)'}{\sin x}\right] \mathrm{d}x = -\ln|\cos x| + \ln|\sin x| + c = \ln\left|\frac{\sin x}{\cos x}\right| + c = \ln|\operatorname{tg} x| + c,$$

$$\text{for } x \in R - \left\{k\frac{\pi}{2};\ k \in Z\right\}.$$

$$\int \frac{x\,\mathrm{d}x}{x^2 - a^2} = \frac{1}{2}\int \frac{2x\,\mathrm{d}x}{x^2 - a^2} = \frac{1}{2}\ln\left|x^2 - a^2\right| + c, \qquad \text{for } a \in R,\ x \in R - \{\pm a\}.$$

$$\int \frac{x\,\mathrm{d}x}{x^2 + a^2} = \frac{1}{2}\int \frac{2x\,\mathrm{d}x}{x^2 + a^2} = \frac{1}{2}\ln\left|x^2 + a^2\right| + c = \frac{1}{2}\ln\left(x^2 + a^2\right) + c, \qquad \text{for } a \in R,\ a \neq 0,\ x \in R.$$

$$\int \frac{\mathrm{d}x}{x} = \int \frac{x\,\mathrm{d}x}{x^2 - 0} = \frac{1}{2}\ln\left|x^2\right| + c = \frac{2}{2}\ln|x| + c = \ln|x| + c, \qquad \text{for } x \in R - \{0\}.$$

$$\int \frac{x\,\mathrm{d}x}{(x^2 - a)^n} = \frac{1}{2}\int \frac{2x\,\mathrm{d}x}{(x^2 - a)^n} = \boxed{\begin{array}{c} x^2 - a = t \\ 2x\,\mathrm{d}x = \mathrm{d}t \end{array}} = \frac{1}{2}\int \frac{\mathrm{d}t}{t^n} = \frac{1}{2}\int t^{-n}\,\mathrm{d}t = \frac{t^{1-n}}{2(1-n)} + c = \frac{(x^2 - a)^{1-n}}{2(1-n)} + c,$$

$$\text{for } n \in N,\ a \geq 0,\ x \in R - \{\pm\sqrt{a}\},\ a \geq 0, \quad \text{resp. for } a < 0,\ x \in R.$$

$$\int \frac{\ln\cos x}{\sin^2 x}\,\mathrm{d}x = \boxed{\begin{array}{l} u = \ln\cos x \ \Rightarrow\ u' = \dfrac{-\sin x}{\cos x} \\ v' = \dfrac{1}{\sin^2 x} \ \Rightarrow\ v = -\operatorname{cotg} x = -\dfrac{\cos x}{\sin x} \end{array}} = -\operatorname{cotg} x \ln\cos x - \int \mathrm{d}x = -\operatorname{cotg} x \ln\cos x - x + c,$$

$$\text{for } x \in \bigcup_{k \in Z}\left[\left\langle -\frac{\pi}{2} + 2k\pi;\ 2k\pi\right) \cup \left(2k\pi;\ \frac{\pi}{2} + 2k\pi\right\rangle\right].$$

$$\int \frac{\mathrm{d}x}{x^6+1} = \int \left[\frac{\frac{1}{3}}{x^2+1} + \frac{-\frac{x}{2\sqrt{3}}+\frac{1}{3}}{x^2-\sqrt{3}x+1} + \frac{\frac{x}{2\sqrt{3}}+\frac{1}{3}}{x^2+\sqrt{3}x+1} \right] \mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}x}{x^2+1} + \frac{1}{2\sqrt{3}} \int \left[\frac{x+\frac{2\sqrt{3}}{3}}{x^2+\sqrt{3}x+1} - \frac{x-\frac{2\sqrt{3}}{3}}{x^2-\sqrt{3}x+1} \right] \mathrm{d}x =$$

$$\boxed{\frac{1}{x^6+1} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-\sqrt{3}x+1} + \frac{Ex+F}{x^2+\sqrt{3}x+1} \implies A=0,\ B=\frac{1}{3},\ C=-\frac{1}{2\sqrt{3}},\ D=\frac{1}{3},\ E=\frac{1}{2\sqrt{3}},\ F=\frac{1}{3}}$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \int \left[\frac{2x+\frac{4\sqrt{3}}{3}}{x^2+\sqrt{3}x+1} - \frac{2x-\frac{4\sqrt{3}}{3}}{x^2-\sqrt{3}x+1} \right] \mathrm{d}x =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \int \left[\frac{2x+\frac{4\sqrt{3}}{3}+\sqrt{3}-\sqrt{3}}{x^2+\sqrt{3}x+1} - \frac{2x-\frac{4\sqrt{3}}{3}-\sqrt{3}+\sqrt{3}}{x^2-\sqrt{3}x+1} \right] \mathrm{d}x =$$

$$\boxed{x^2 \pm \sqrt{3}x + 1 = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 + 1 - \frac{3}{4} = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{4} = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 > 0}$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \int \left[\frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1} - \frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1} \right] \mathrm{d}x + \frac{1}{4\sqrt{3}} \int \left[\frac{\frac{\sqrt{3}}{3}}{\left(x+\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} - \frac{-\frac{\sqrt{3}}{3}}{\left(x-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \right] \mathrm{d}x =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \int \frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}\,\mathrm{d}x - \frac{1}{4\sqrt{3}} \int \frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1}\,\mathrm{d}x + \frac{1}{12}\int \frac{\mathrm{d}x}{\left(x+\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} + \frac{1}{12}\int \frac{\mathrm{d}x}{\left(x-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} =$$

$$\boxed{x+\frac{\sqrt{3}}{2} = t,\ \mathrm{d}x = \mathrm{d}t, \qquad x-\frac{\sqrt{3}}{2} = u,\ \mathrm{d}x = \mathrm{d}u}$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \ln\left|x^2+\sqrt{3}x+1\right| - \frac{1}{4\sqrt{3}} \ln\left|x^2-\sqrt{3}x+1\right| + \frac{1}{12}\int \frac{\mathrm{d}t}{t^2+\left(\frac{1}{2}\right)^2} + \frac{1}{12}\int \frac{\mathrm{d}u}{u^2+\left(\frac{1}{2}\right)^2} =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \ln\left(x^2+\sqrt{3}x+1\right) - \frac{1}{4\sqrt{3}} \ln\left(x^2-\sqrt{3}x+1\right) + \frac{2}{12}\operatorname{arctg} 2t + \frac{2}{12}\operatorname{arctg} 2u + c =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}} \ln \frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1} + \frac{1}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) + \frac{1}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) + c, \qquad \text{for } x \in R.$$

$$\int \frac{x\,\mathrm{d}x}{x^6+1} = \boxed{\begin{matrix} x^2=t \\ 2x\,\mathrm{d}x=\mathrm{d}t \end{matrix}} = \frac{1}{2}\int \frac{\mathrm{d}t}{t^3+1} = \frac{1}{2}\int \frac{\mathrm{d}t}{(t+1)(t^2-t+1)} = \frac{1}{2}\int \left[\frac{\frac{1}{3}}{t+1} + \frac{-\frac{t}{3}+\frac{2}{3}}{t^2-t+1}\right]\mathrm{d}t =$$

$$\boxed{\frac{1}{t^3+1} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1} \implies 1 = A(t^2-t+1)+(Bt+C)(t+1) \implies A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{2}{3}}$$

$$= \frac{1}{6}\int \frac{\mathrm{d}t}{t+1} - \frac{1}{6}\int \frac{t-2}{t^2-t+1}\,\mathrm{d}t = \frac{1}{6}\int \frac{\mathrm{d}t}{t+1} - \frac{1}{12}\int \frac{2t-4}{t^2-t+1}\,\mathrm{d}t = \frac{1}{6}\int \frac{\mathrm{d}t}{t+1} - \frac{1}{12}\int \frac{2t-1-3}{t^2-t+1}\,\mathrm{d}t =$$

$$\boxed{t^2-t+1 = \left(t-\frac{1}{2}\right)^2 + 1 - \frac{1}{4} = \left(t-\frac{1}{2}\right)^2 + \frac{3}{4} = \left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 > 0}$$

$$= \frac{1}{6}\ln|t+1| - \frac{1}{12}\int \frac{2t-1}{t^2-t+1}\,\mathrm{d}t + \frac{1}{12}\int \frac{3\,\mathrm{d}t}{t^2-t+1} = \frac{1}{12}\ln(t+1)^2 - \frac{1}{12}\ln\left|t^2-t+1\right| + \frac{1}{4}\int \frac{\mathrm{d}t}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} =$$

$$= \boxed{t-\frac{1}{2}=z,\ \mathrm{d}t=\mathrm{d}z} = \frac{1}{12}\ln(t+1)^2 - \frac{1}{12}\ln\left(t^2-t+1\right) + \frac{1}{4}\int \frac{\mathrm{d}z}{z^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{1}{12}\ln\frac{(t+1)^2}{t^2-t+1} + \frac{1}{4}\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2z}{\sqrt{3}} + c_1 =$$

$$= \frac{1}{12}\ln\frac{t^2+2t+1}{t^2-t+1} + \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + c_1 = \frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \frac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + c_1, \qquad \text{for } x\in R.$$

$$\int \frac{x\,\mathrm{d}x}{x^6+1} = \int \left[\frac{\frac{x}{3}}{x^2+1} + \frac{-\frac{x}{6}+\frac{\sqrt{3}}{6}}{x^2-\sqrt{3}x+1} + \frac{-\frac{x}{6}-\frac{\sqrt{3}}{6}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x = \frac{1}{6}\int \frac{2x\,\mathrm{d}x}{x^2+1} - \frac{1}{12}\int \left[\frac{2x-2\sqrt{3}}{x^2-\sqrt{3}x+1} + \frac{2x+2\sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x =$$

$$\boxed{\frac{x}{x^6+1} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-\sqrt{3}x+1} + \frac{Ex+F}{x^2+\sqrt{3}x+1} \implies A=\frac{1}{3},\ B=0,\ C=-\frac{1}{6},\ D=\frac{\sqrt{3}}{6},\ E=-\frac{1}{6},\ F=-\frac{\sqrt{3}}{6}}$$

$$= \frac{1}{6}\ln\left|x^2+1\right| - \frac{1}{12}\int \left[\frac{2x-\sqrt{3}-\sqrt{3}}{x^2-\sqrt{3}x+1} + \frac{2x+\sqrt{3}+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x = \boxed{x^2\pm\sqrt{3}x+1 = \left(x\pm\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 > 0} =$$

$$= \frac{1}{12}\ln\left(x^2+1\right)^2 - \frac{1}{12}\int \left[\frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1} + \frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x + \frac{1}{12}\int \left[\frac{\sqrt{3}}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2} - \frac{\sqrt{3}}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right]\mathrm{d}x =$$

$$\boxed{x-\frac{\sqrt{3}}{2}=t,\ \mathrm{d}x=\mathrm{d}t, \qquad x+\frac{\sqrt{3}}{2}=u,\ \mathrm{d}x=\mathrm{d}u} \qquad \boxed{\left(x^2-\sqrt{3}x+1\right)\left(x^2+\sqrt{3}x+1\right) = x^4-x^2+1}$$

$$= \frac{1}{12}\ln\left(x^4+2x^2+1\right) - \frac{1}{12}\ln\left|x^2-\sqrt{3}x+1\right| - \frac{1}{12}\ln\left|x^2+\sqrt{3}x+1\right| + \frac{\sqrt{3}}{12}\int \frac{\mathrm{d}t}{t^2+\left(\frac{1}{2}\right)^2} - \frac{\sqrt{3}}{12}\int \frac{\mathrm{d}u}{u^2+\left(\frac{1}{2}\right)^2} =$$

$$= \frac{1}{12}\ln\left(x^4+2x^2+1\right) - \frac{1}{12}\ln\left(x^2-\sqrt{3}x+1\right) - \frac{1}{12}\ln\left(x^2+\sqrt{3}x+1\right) + \frac{2\sqrt{3}}{12}\operatorname{arctg}2t - \frac{2\sqrt{3}}{12}\operatorname{arctg}2u + c_2 =$$

$$= \frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1} + \frac{\sqrt{3}}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) - \frac{\sqrt{3}}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) + c_2, \qquad \text{for } x\in R.$$

$$??\quad \frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}}+c_1 \stackrel{?}{=} \frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\left[\operatorname{arctg}\left(2x-\sqrt{3}\right)-\operatorname{arctg}\left(2x+\sqrt{3}\right)\right]+c_2\quad ??$$

$$\left[\frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}}+c_1\right]-\left[\frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right)-\frac{\sqrt{3}}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right)+c_2\right]=$$

$$=c_1-c_2+\frac{\sqrt{3}}{6}\left[\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}}-\operatorname{arctg}\left(2x-\sqrt{3}\right)+\operatorname{arctg}\left(2x+\sqrt{3}\right)\right]=\boxed{\operatorname{arctg}\alpha=\frac{\pi}{2}-\operatorname{arctg}\alpha\ \text{ for } \alpha\in R}=$$

$$=c_1-c_2+\frac{\sqrt{3}}{6}\left[\frac{\pi}{2}-\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}-\frac{\pi}{2}+\operatorname{arccotg}\left(2x-\sqrt{3}\right)+\frac{\pi}{2}-\operatorname{arccotg}\left(2x+\sqrt{3}\right)\right]=\boxed{c_1-c_2+\frac{\pi\sqrt{3}}{2\cdot 6}=c}=$$

$$=c+\frac{\sqrt{3}}{6}\left[\operatorname{arccotg}\left(2x-\sqrt{3}\right)-\operatorname{arccotg}\left(2x+\sqrt{3}\right)-\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}\right]=\boxed{\operatorname{arccotg}\alpha-\operatorname{arccotg}\beta=\operatorname{arccotg}\frac{\alpha\beta+1}{\beta-\alpha}\ \text{ for } \alpha,\beta\in R, \alpha\neq\beta}=$$

$$=c+\frac{\sqrt{3}}{6}\left[\operatorname{arccotg}\frac{(2x-\sqrt{3})\,(2x+\sqrt{3})+1}{(2x+\sqrt{3})-(2x-\sqrt{3})}-\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}\right]=c+\frac{\sqrt{3}}{6}\left[\operatorname{arccotg}\frac{4x^2-3+1}{2\sqrt{3}}-\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}\right]=$$

$$=c+\frac{\sqrt{3}}{6}\left[\operatorname{arccotg}\frac{4x^2-2}{2\sqrt{3}}-\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}\right]=c+\frac{\sqrt{3}}{6}\left[\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}-\operatorname{arccotg}\frac{2x^2-1}{\sqrt{3}}\right]=c=c_1-c_2+\frac{\pi\sqrt{3}}{12}\ \text{(i.e. const.)}.$$

$$\Longrightarrow \int\frac{x\,\mathrm{d}x}{x^6+1}=\frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}}+c_1=$$

$$=\frac{1}{12}\ln\frac{x^4+2x^2+1}{x^4-x^2+1}+\frac{\sqrt{3}}{6}\left[\operatorname{arctg}\left(2x-\sqrt{3}\right)-\operatorname{arctg}\left(2x+\sqrt{3}\right)\right]+c_1-\frac{\pi\sqrt{3}}{12},\qquad \text{for } x\in R.$$

$$\int\frac{x^2\,\mathrm{d}x}{x^6+1}=\frac{1}{3}\int\frac{3x^2\,\mathrm{d}x}{x^6+1}=\boxed{\begin{matrix}x^3=t\\ 3x^2\,\mathrm{d}x=\mathrm{d}t\end{matrix}}=\frac{1}{3}\int\frac{\mathrm{d}t}{t^2+1}=\frac{1}{3}\operatorname{arctg}t+c_1=\frac{1}{3}\operatorname{arctg}x^3+c_1,\qquad \text{for } x\in R.$$

$$\int\frac{x^2\,\mathrm{d}x}{x^6+1}=\int\left[\frac{-\frac{1}{3}}{x^2+1}+\frac{\frac{1}{6}}{x^2-\sqrt{3}x+1}+\frac{\frac{1}{6}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x=-\frac{1}{3}\int\frac{\mathrm{d}x}{x^2+1}+\frac{1}{6}\int\frac{\mathrm{d}x}{x^2-\sqrt{3}x+1}+\frac{1}{6}\int\frac{\mathrm{d}x}{x^2+\sqrt{3}x+1}=$$

$$\boxed{\frac{x^2}{x^6+1}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2-\sqrt{3}x+1}+\frac{Ex+F}{x^2+\sqrt{3}x+1}\ \Longrightarrow\ A=0,\ B=-\frac{1}{3},\ C=0,\ D=\frac{1}{6},\ E=0,\ F=\frac{1}{6}}$$

$$=\boxed{x^2\pm\sqrt{3}x+1=\left(x\pm\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2>0}=\frac{1}{6}\int\frac{\mathrm{d}x}{\left(x-\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}+\frac{1}{6}\int\frac{\mathrm{d}x}{\left(x+\frac{\sqrt{3}}{2}\right)^2+\left(\frac{1}{2}\right)^2}-\frac{1}{3}\operatorname{arctg}x=$$

$$=\boxed{x-\frac{\sqrt{3}}{2}=t,\ \mathrm{d}x=\mathrm{d}t,\qquad x+\frac{\sqrt{3}}{2}=u,\ \mathrm{d}x=\mathrm{d}u}=\frac{1}{6}\int\frac{\mathrm{d}t}{t^2+\left(\frac{1}{2}\right)^2}+\frac{1}{6}\int\frac{\mathrm{d}u}{u^2+\left(\frac{1}{2}\right)^2}-\frac{1}{3}\operatorname{arctg}x=$$

$$=\frac{2}{6}\operatorname{arctg}2t+\frac{2}{6}\operatorname{arctg}2u-\frac{1}{3}\operatorname{arctg}x+c_2=\frac{1}{3}\operatorname{arctg}\left(2x-\sqrt{3}\right)+\frac{1}{3}\operatorname{arctg}\left(2x+\sqrt{3}\right)-\frac{1}{3}\operatorname{arctg}x+c_2,\qquad \text{for } x\in R.$$

$$??\quad \frac{1}{3}\operatorname{arctg} x^3 + c_1 \overset{?}{=\!=} \frac{1}{3}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \frac{1}{3}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \frac{1}{3}\operatorname{arctg} x + c_2 \quad ??$$

$$E(x) = \left[\frac{1}{3}\operatorname{arctg} x^3 + c_1\right] - \left[\frac{1}{3}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \frac{1}{3}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \frac{1}{3}\operatorname{arctg} x + c_2\right] = \boxed{\operatorname{arctg}\alpha = \frac{\pi}{2} - \operatorname{arctg}\alpha \ \text{ for } \alpha\in R} =$$

$$= \frac{1}{3}\left[\frac{\pi}{2} - \operatorname{arccotg} x^3 - \frac{\pi}{2} + \operatorname{arccotg}\left(2x-\sqrt{3}\right) - \frac{\pi}{2} + \operatorname{arccotg}\left(2x+\sqrt{3}\right) + \frac{\pi}{2} - \operatorname{arccotg} x\right] + c_1 - c_2 =$$

$$= \frac{1}{3}\left[\operatorname{arccotg}\left(2x-\sqrt{3}\right) + \operatorname{arccotg}\left(2x+\sqrt{3}\right) - \operatorname{arccotg} x - \operatorname{arccotg} x^3\right] + c_1 - c_2 =$$

$$\boxed{\operatorname{arccotg}\alpha + \operatorname{arccotg}\beta = \operatorname{arccotg}\frac{\alpha\beta-1}{\alpha+\beta} \ \text{ for } \alpha,\beta\in R, \alpha\neq-\beta} \quad \boxed{2x-\sqrt{3}\neq-\left(2x+\sqrt{3}\right) \implies x\neq\frac{\sqrt{3}}{2}} \quad \boxed{x\neq -x^3 \implies x\neq 0}$$

$$= \frac{1}{3}\left[\operatorname{arccotg}\frac{\left(2x-\sqrt{3}\right)\left(2x+\sqrt{3}\right)-1}{\left(2x-\sqrt{3}\right)+\left(2x+\sqrt{3}\right)} - \operatorname{arccotg}\frac{x\cdot x^3-1}{x+x^3}\right] + c_1 - c_2 = \frac{1}{3}\left[\operatorname{arccotg}\frac{4x^2-3-1}{4x} - \operatorname{arccotg}\frac{x^4-1}{x+x^3}\right] + c_1 - c_2 =$$

$$= \frac{1}{3}\left[\operatorname{arccotg}\frac{x^2-1}{x} - \operatorname{arccotg}\frac{(x^2-1)(x^2+1)}{x(1+x^2)}\right] + c_1 - c_2 = \frac{1}{3}\left[\operatorname{arccotg}\frac{x^2-1}{x} - \operatorname{arccotg}\frac{x^2-1}{x}\right] + c_1 - c_2 = c_1 - c_2 \text{ (i.e. const.).}$$

$$\text{for } \ x\in R - \left\{0, \frac{\sqrt{3}}{2}\right\}.$$

$$E(0) = \left[\frac{1}{3}\operatorname{arctg} 0^3 + c_1\right] - \left[\frac{1}{3}\operatorname{arctg}\left(2\cdot 0-\sqrt{3}\right) + \frac{1}{3}\operatorname{arctg}\left(2\cdot 0+\sqrt{3}\right) - \frac{1}{3}\operatorname{arctg} 0 + c_2\right] =$$

$$= \frac{1}{3}\operatorname{arctg} 0 + c_1 - \frac{1}{3}\operatorname{arctg}\left(-\sqrt{3}\right) - \frac{1}{3}\operatorname{arctg}\sqrt{3} + \frac{1}{3}\operatorname{arctg} 0 - c_2 = c_1 - c_2, \qquad \text{for } \ x = 0.$$

$$E\left(\frac{\sqrt{3}}{2}\right) = \left[\frac{1}{3}\operatorname{arctg}\left(\frac{\sqrt{3}}{2}\right)^3 + c_1\right] - \left[\frac{1}{3}\operatorname{arctg}\left(2\frac{\sqrt{3}}{2}-\sqrt{3}\right) + \frac{1}{3}\operatorname{arctg}\left(2\frac{\sqrt{3}}{2}+\sqrt{3}\right) - \frac{1}{3}\operatorname{arctg}\frac{\sqrt{3}}{2} + c_2\right] =$$

$$= \frac{1}{3}\operatorname{arctg}\frac{3\sqrt{3}}{8} + c_1 - \frac{1}{3}\operatorname{arctg} 0 - \frac{1}{3}\operatorname{arctg} 2\sqrt{3} + \frac{1}{3}\operatorname{arctg}\frac{\sqrt{3}}{2} - c_2 = c_1 - c_2 + \frac{1}{3}\left[\operatorname{arctg}\frac{3\sqrt{3}}{8} + \operatorname{arctg}\frac{\sqrt{3}}{2} - \operatorname{arctg} 2\sqrt{3}\right] =$$

$$= \boxed{\operatorname{arctg}\alpha - \operatorname{arctg}\beta = \operatorname{arctg}\frac{\alpha-\beta}{1+\alpha\beta} \ \text{ for } \alpha,\beta\in R, \alpha\beta>-1} = c_1 - c_2 + \frac{1}{3}\left[\operatorname{arctg}\frac{3\sqrt{3}}{8} + \operatorname{arctg}\frac{\frac{\sqrt{3}}{2}-2\sqrt{3}}{1+\frac{\sqrt{3}}{2}2\sqrt{3}}\right] =$$

$$= c_1 - c_2 + \frac{1}{3}\left[\operatorname{arctg}\frac{3\sqrt{3}}{8} + \operatorname{arctg}\frac{-\frac{3\sqrt{3}}{2}}{1+3}\right] = c_1 - c_2 + \frac{1}{3}\left[\operatorname{arctg}\frac{3\sqrt{3}}{8} + \operatorname{arctg}\frac{-3\sqrt{3}}{8}\right] = c_1 - c_2, \qquad \text{for } \ x = \frac{\sqrt{3}}{2}.$$

$$\implies \int\frac{x^2\,dx}{x^6+1} = \frac{1}{3}\operatorname{arctg} x^3 + c_1 =$$

$$= \frac{1}{3}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \frac{1}{3}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \frac{1}{3}\operatorname{arctg} x + c_1, \qquad \text{for } \ x\in R.$$

$$\int \frac{x^3\,\mathrm{d}x}{x^6+1} = \frac{1}{2}\int \frac{2x\cdot x^2}{x^6+1}\,\mathrm{d}x = \boxed{\begin{matrix} x^2 = t \\ 2x\,\mathrm{d}x = \mathrm{d}t \end{matrix}} = \frac{1}{2}\int \frac{t\,\mathrm{d}t}{t^3+1} = \frac{1}{2}\int \frac{t\,\mathrm{d}t}{(t+1)(t^2-t+1)} = \frac{1}{2}\int \left[\frac{-\frac{1}{3}}{t+1} + \frac{\frac{t}{3}+\frac{1}{3}}{t^2-t+1}\right]\mathrm{d}t =$$

$$\boxed{\frac{t}{t^3+1} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1} \implies t = A(t^2-t+1) + (Bt+C)(t+1) \implies A = -\frac{1}{3},\ B = \frac{1}{3},\ C = \frac{1}{3}}$$

$$= \frac{1}{6}\int \frac{t+1}{t^2-t+1}\,\mathrm{d}t - \frac{1}{6}\int \frac{\mathrm{d}t}{t+1} = \frac{1}{12}\int \frac{2t+2}{t^2-t+1}\,\mathrm{d}t - \frac{1}{6}\int \frac{\mathrm{d}t}{t+1} = \frac{1}{12}\int \frac{2t-1+3}{t^2-t+1}\,\mathrm{d}t - \frac{1}{6}\int \frac{\mathrm{d}t}{t+1} =$$

$$\boxed{t^2-t+1 = \left(t-\frac{1}{2}\right)^2 + 1 - \frac{1}{4} = \left(t-\frac{1}{2}\right)^2 + \frac{3}{4} = \left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 > 0}$$

$$= \frac{1}{12}\int \frac{2t-1}{t^2-t+1}\,\mathrm{d}t + \frac{1}{4}\int \frac{\mathrm{d}t}{(t-\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{6}\ln|t+1| = \boxed{t-\frac{1}{2} = z,\ \mathrm{d}t = \mathrm{d}z} =$$

$$= \frac{1}{12}\ln\left(t^2-t+1\right) + \frac{1}{4}\int \frac{\mathrm{d}z}{z^2 + \left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{12}\ln(t+1)^2 = \frac{1}{12}\ln\frac{t^2-t+1}{(t+1)^2} + \frac{1}{4}\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2z}{\sqrt{3}} + c =$$

$$= \frac{1}{12}\ln\frac{t^2-t+1}{t^2+2t+1} + \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + c = \frac{1}{12}\ln\frac{x^4-x^2+1}{x^4+2x^2+1} + \frac{1}{2\sqrt{3}}\operatorname{arctg}\frac{2x^2-1}{\sqrt{3}} + c, \qquad \text{for } x\in R.$$

$$\int \frac{x^4\,\mathrm{d}x}{x^6+1} = \int \left[\frac{\frac{1}{3}}{x^2+1} + \frac{\frac{x}{2\sqrt{3}} - \frac{1}{6}}{x^2-\sqrt{3}x+1} + \frac{-\frac{x}{2\sqrt{3}} - \frac{1}{6}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x = \frac{1}{3}\int \frac{\mathrm{d}x}{x^2+1} + \frac{1}{2\sqrt{3}}\int \left[\frac{x-\frac{\sqrt{3}}{3}}{x^2-\sqrt{3}x+1} - \frac{x+\frac{\sqrt{3}}{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x =$$

$$\boxed{\frac{x^4}{x^6+1} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2-\sqrt{3}x+1} + \frac{Ex+F}{x^2+\sqrt{3}x+1} \implies A=0,\ B=\frac{1}{3},\ C=\frac{1}{2\sqrt{3}},\ D=-\frac{1}{6},\ E=-\frac{1}{2\sqrt{3}},\ F=-\frac{1}{6}}$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}}\int \left[\frac{2x - \frac{2\sqrt{3}}{3} + \sqrt{3} - \sqrt{3}}{x^2-\sqrt{3}x+1} - \frac{2x + \frac{2\sqrt{3}}{3} - \sqrt{3} + \sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x =$$

$$\boxed{x^2 \pm \sqrt{3}x + 1 = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 + 1 - \frac{3}{4} = \left(x \pm \frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 > 0} \quad \boxed{x - \frac{\sqrt{3}}{2} = t,\ \mathrm{d}x = \mathrm{d}t} \quad \boxed{x + \frac{\sqrt{3}}{2} = u,\ \mathrm{d}x = \mathrm{d}u}$$

$$= \frac{\operatorname{arctg} x}{3} + \frac{1}{4\sqrt{3}}\int \left[\frac{2x-\sqrt{3}}{x^2-\sqrt{3}x+1} - \frac{2x+\sqrt{3}}{x^2+\sqrt{3}x+1}\right]\mathrm{d}x + \frac{1}{4\sqrt{3}}\int \left[\frac{\frac{\sqrt{3}}{3}}{\left(x-\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} - \frac{-\frac{\sqrt{3}}{3}}{\left(x+\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2}\right]\mathrm{d}x =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}}\ln\left|x^2-\sqrt{3}x+1\right| - \frac{1}{4\sqrt{3}}\ln\left|x^2+\sqrt{3}x+1\right| + \frac{1}{12}\int \frac{\mathrm{d}t}{t^2+\left(\frac{1}{2}\right)^2} + \frac{1}{12}\int \frac{\mathrm{d}u}{u^2+\left(\frac{1}{2}\right)^2} =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}}\ln\left(x^2-\sqrt{3}x+1\right) - \frac{1}{4\sqrt{3}}\ln\left(x^2+\sqrt{3}x+1\right) + \frac{2}{12}\operatorname{arctg} 2t + \frac{2}{12}\operatorname{arctg} 2u + c =$$

$$= \frac{1}{3}\operatorname{arctg} x + \frac{1}{4\sqrt{3}}\ln\frac{x^2-\sqrt{3}x+1}{x^2+\sqrt{3}x+1} + \frac{1}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) + \frac{1}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) + c, \qquad \text{for } x\in R.$$

$$\int \frac{x^5\,dx}{x^6+1} = \frac{1}{6}\int \frac{6x^5}{x^6+1}\,dx = \boxed{\begin{matrix} x^6 = t \\ 6x^5\,dx = dt \end{matrix}} = \frac{1}{3}\int \frac{dt}{t+1} = \frac{1}{6}\ln|t+1| + c = \frac{1}{6}\ln\left|x^6+1\right| + c = \frac{1}{6}\ln\left(x^6+1\right) + c, \qquad \text{for } x\in R.$$

$$\int \frac{x^6\,dx}{x^6+1} = \int \frac{x^6+1-1}{x^6+1}\,dx = \int \left[1 - \frac{1}{x^6+1}\right] dx = \int dx - \int \frac{dx}{x^6+1} =$$

$$= x - \frac{1}{3}\operatorname{arctg} x - \frac{1}{4\sqrt{3}}\ln\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1} - \frac{1}{6}\operatorname{arctg}\left(2x+\sqrt{3}\right) - \frac{1}{6}\operatorname{arctg}\left(2x-\sqrt{3}\right) + c, \qquad \text{for } x\in R.$$

$$\int \frac{dx}{x\ln x} = \int \frac{\frac{1}{x}}{\ln x}\,dx = \int \frac{[\ln x]'}{\ln x}\,dx = \ln|\ln x| + c, \qquad \text{for } x>0.$$

$$\int x^2\ln x\,dx = \boxed{\begin{matrix} u=\ln x & \Rightarrow & u'=\frac{1}{x} \\ v'=x^2 & \Rightarrow & v=\frac{x^3}{3} \end{matrix}} = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + c, \qquad \text{for } x>0.$$

$$\int \frac{\ln x}{\sqrt{x}}\,dx = \boxed{\begin{matrix} u=\ln x & \Rightarrow & u'=\frac{1}{x} \\ v'=x^{-\frac{1}{2}} & \Rightarrow & v=2x^{\frac{1}{2}} \end{matrix}} = 2\sqrt{x}\ln x - 2\int \frac{x^{\frac{1}{2}}}{x}\,dx = 2\sqrt{x}\ln x - 2\int x^{-\frac{1}{2}}\,dx = 2\sqrt{x}\ln x - 2\cdot 2x^{\frac{1}{2}} + c = 2\sqrt{x}\ln x - 4\sqrt{x} + c,$$

for $x>0$.

$$\int \frac{dx}{\sqrt{x(1-x)}} = \int \frac{t^2+1}{t}\,\frac{2t\,dt}{(t^2+1)^2} = \int \frac{2\,dt}{t^2+1} = 2\operatorname{arctg} t + c_1 = 2\operatorname{arctg}\sqrt{\frac{x}{1-x}} + c_1, \quad \text{for } x\in(0\,;1).$$

$$x(1-x) = x - x^2 > 0 \implies x > x^2 \implies x\in(0\,;1)$$

3-rd Euler: $t = \sqrt{\dfrac{x}{1-x}} \implies t^2 = \dfrac{x}{1-x} \implies t^2 - t^2x = x \implies x = \dfrac{t^2}{t^2+1} \implies dx = \dfrac{2t(t^2+1)-t^2\cdot 2t}{(t^2+1)^2}\,dt = \dfrac{2t\,dt}{(t^2+1)^2}$

$x\in(0\,;1) \implies t\in(0\,;\infty)$, $\quad x(1-x) = \dfrac{t^2}{t^2+1}\left(1-\dfrac{t^2}{t^2+1}\right) = \dfrac{t^2(t^2+1-t^2)}{(t^2+1)^2} = \dfrac{t^2}{(t^2+1)^2} \implies \sqrt{x(1-x)} = \dfrac{t}{t^2+1}$

$$\int \frac{dx}{\sqrt{x(1-x)}} = \boxed{t=\frac{1}{x} \Rightarrow x=\frac{1}{t} \Rightarrow dx = -\frac{dt}{t^2}} = -\int \frac{1}{\sqrt{\frac{1}{t}\left(1-\frac{1}{t}\right)}}\frac{dt}{t^2} = -\int \frac{dt}{t\sqrt{t-1}} = \cdots$$

☹

$$\int \frac{dx}{\sqrt{x(1-x)}} = \boxed{x(1-x) = -[x^2-x] = -\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}\right] \implies x-\frac{1}{2} = t \implies dx = dt} = \int \frac{dt}{\sqrt{\left(\frac{1}{2}\right)^2 - t^2}} = \arcsin 2t + c_2 =$$

$$= \arcsin(2x-1) + c_2, \qquad \text{for } x\in(0\,;1).$$

$$\int \frac{\mathrm{d}x}{\left(x-\sqrt{x^2-1}\right)^2} = \int \frac{1}{\left(x-\sqrt{x^2-1}\right)^2} \frac{\left(x+\sqrt{x^2-1}\right)^2}{\left(x+\sqrt{x^2-1}\right)^2}\,\mathrm{d}x = \int \frac{x^2+2x\sqrt{x^2-1}+[x^2-1]}{(x^2-[x^2-1])^2}\,\mathrm{d}x = \int \frac{2x^2-1+2x\sqrt{x^2-1}}{1^2}\,\mathrm{d}x =$$

$$= \int (2x^2-1)\,\mathrm{d}x + \int 2x\sqrt{x^2-1}\,\mathrm{d}x = \boxed{x^2-1=t \ \Rightarrow\ 2x\,\mathrm{d}x=\mathrm{d}t} = \int (2x^2-1)\,\mathrm{d}x + \int t^{\frac{1}{2}}\,\mathrm{d}t =$$

$$= \frac{2}{3}x^3 - x + \frac{2}{3}t^{\frac{3}{2}} + c = \frac{2}{3}x^3 - x + \frac{2}{3}\left(x^2-1\right)^{\frac{3}{2}} + c = \frac{2}{3}x^3 - x + \frac{2}{3}\sqrt{(x^2-1)^3} + c, \qquad \text{for } x\in R-(-1\,;\,1).$$

$$\int \frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{1-x^2}} + \int \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = \arcsin x - \frac{1}{2}\int \frac{\mathrm{d}t}{\sqrt{t}} = \boxed{\begin{array}{c} t=1-x^2 \\ \mathrm{d}t=-2x\,\mathrm{d}x \ \Rightarrow\ x\,\mathrm{d}x=-\,\mathrm{d}t/2 \end{array}} = \arcsin x - \frac{1}{2}\int t^{-\frac{1}{2}}\,\mathrm{d}t =$$

$$= \arcsin x - \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + c = \arcsin x - \sqrt{t} + c = \arcsin x - \sqrt{1-x^2} + c, \quad \text{for } x\in(0\,;\,1).$$

$$\int \frac{1+x}{\sqrt{1-x^2}}\,\mathrm{d}x = \int \frac{1+x}{\sqrt{(1-x)(1+x)}}\,\mathrm{d}x = \int \sqrt{\frac{1+x}{1-x}}\,\mathrm{d}x = \int \frac{4t^2\,\mathrm{d}t}{(t^2+1)^2} = \int \frac{2t\cdot 2t\,\mathrm{d}t}{(t^2+1)^2} = \boxed{\begin{array}{ll} u=2t & \Rightarrow\ u'=2 \\ v'=\dfrac{2t}{(t^2+1)^2} & \Rightarrow\ v=-\dfrac{1}{t^2+1} \end{array}} =$$

$$\boxed{\sqrt{\frac{1+x}{1-x}}=t \ \Rightarrow\ \frac{1+x}{1-x}=t^2 \ \Rightarrow\ 1+x=t^2-t^2x \ \Rightarrow\ x=\frac{t^1-1}{t^2+1} \ \Rightarrow\ \mathrm{d}x=\frac{2t(t^2+1)-(t^2-1)2t}{(t^2+1)^2}\,\mathrm{d}t=\frac{4t\,\mathrm{d}t}{(t^2+1)^2}}$$

$$= -\frac{2t}{t^2+1} - \int \frac{-2\,\mathrm{d}t}{t^2+1} = 2\int \frac{\mathrm{d}t}{t^2+1} - \frac{2t}{t^2+1} = 2\,\mathrm{arctg}\,t - \frac{2t}{t^2+1} + c = 2\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\dfrac{1+x}{1-x}}}{\dfrac{1+x}{1-x}+1} + c =$$

$$= 2\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}} - \frac{2\sqrt{\dfrac{1+x}{1-x}}}{\dfrac{2}{1-x}} + c = 2\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c = 2\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c, \qquad \text{for } x\in(0\,;\,1).$$

$$\int \frac{x^n}{x-1}\,\mathrm{d}x = \int \frac{x^n-1+1}{x-1}\,\mathrm{d}x = \int \frac{x^n-1}{x-1}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{x-1} = \int \left[x^{n-1}+x^{n-2}+\cdots+x+1\right]\mathrm{d}x + \int \frac{\mathrm{d}x}{x-1} =$$

$$= \frac{x^n}{n} + \frac{x^{n-1}}{n-1} + \cdots + \frac{x^2}{2} + x + \ln|x-1| + c = \ln|x-1| + \sum_{k=1}^{n}\frac{x^k}{k} + c, \qquad \text{for } x\in R-\{1\},\ n\in N.$$

$$\int \frac{x}{x-1}\,\mathrm{d}x = \int \frac{x-1+1}{x-1}\,\mathrm{d}x = \int \mathrm{d}x + \int \frac{\mathrm{d}x}{x-1} = x + \ln|x-1| + c, \qquad \text{for } x\in R-\{1\}.$$

$$\int \frac{x^3}{x-1}\,\mathrm{d}x = \int \frac{x^3-1+1}{x-1}\,\mathrm{d}x = \int \left[x^2+x+1\right]\mathrm{d}x + \int \frac{\mathrm{d}x}{x-1} = \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + c, \qquad \text{for } x\in R-\{1\}.$$

$$\int \frac{x^9}{x-1}\,\mathrm{d}x = \frac{x^9}{9} + \frac{x^8}{8} + \frac{x^7}{7} + \frac{x^6}{6} + \frac{x^5}{5} + \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + c = \ln|x-1| + \sum_{k=1}^{9}\frac{x^k}{k} + c, \qquad \text{for } x\in R-\{1\}.$$

$$\int \frac{dx}{\cos x} = \boxed{t=\operatorname{tg}\frac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2}} = \int \frac{1+t^2}{1-t^2}\frac{2\,dt}{1+t^2} = \int \frac{2\,dt}{1-t^2} = -\int \frac{2\,dt}{t^2-1} = -\frac{2}{2}\ln\left|\frac{t-1}{t+1}\right| + c =$$

$$= -\ln\left|\frac{t-1}{t+1}\right| + c = \ln\left|\frac{t+1}{t-1}\right| + c = \ln\left|\frac{\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}\frac{x}{2}-1}\right| + c, \qquad \text{for } x\in R-\left\{\frac{\pi}{2}+k\pi\,;\,k\in Z\right\}.$$

$$\int \frac{dx}{\cos x} = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{\cos x\,dx}{1-\sin^2 x} = \boxed{\begin{matrix} t=\sin x\\ dt=\cos x\,dx\end{matrix}} = \int \frac{dt}{1-t^2} = -\int \frac{dt}{t^2-1} = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + c =$$

$$= \frac{1}{2}\ln\left|\frac{\sin x+1}{\sin x-1}\right| + c = \frac{1}{2}\ln\frac{1+\sin x}{1-\sin x} + c = \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + c, \qquad \text{for } x\in R-\left\{\frac{\pi}{2}+k\pi\,;\,k\in Z\right\}.$$

$$\int \frac{dx}{1+\cos x} = \boxed{t=\operatorname{tg}\frac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ 1+\cos x=1+\frac{1-t^2}{1+t^2}=\frac{2}{1+t^2}} = \int \frac{1+t^2}{2}\frac{2\,dt}{1+t^2} = \int dt = t + c = \operatorname{tg}\frac{x}{2} + c,$$

$$\text{for } x\in R-\{\pi+2k\pi\,;\,k\in Z\}.$$

$$\int \frac{dx}{1+\cos x} = \int \frac{1-\cos x}{(1+\cos x)(1-\cos x)}\,dx = \int \frac{1-\cos x}{1-\cos^2 x}\,dx = \int \frac{dx}{\sin^2 x} - \int \frac{\cos x}{\sin^2 x}\,dx = \boxed{\begin{matrix}\sin x=t\\ \cos x\,dx=dt\end{matrix}} = -\operatorname{cotg} x - \int \frac{dt}{t^2} =$$

$$= -\operatorname{cotg} x - \frac{t^{-1}}{-1} + c = -\operatorname{cotg} x + \frac{1}{t} + c = -\operatorname{cotg} x + \frac{1}{\sin x} + c = -\frac{\cos x}{\sin x} + \frac{1}{\sin x} + c = \frac{1-\cos x}{\sin x} + c, \qquad \text{for } x\in R-\{k\pi\,;\,k\in Z\}.$$

$$\int \frac{dx}{\sin x} = \boxed{t=\operatorname{tg}\frac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ \sin x=\frac{2t}{1+t^2}} = \int \frac{1+t^2}{2t}\frac{2\,dt}{1+t^2} = \int \frac{dt}{t} = \ln|t| + c = \ln\left|\operatorname{tg}\frac{x}{2}\right| + c, \qquad \text{for } x\in R-\{k\pi\,;\,k\in Z\}.$$

$$\int \frac{dx}{\sin x} = \int \frac{\sin x\,dx}{\sin^2 x} = \int \frac{\sin x\,dx}{1-\cos^2 x} = \boxed{\begin{matrix} t=\cos x\\ dt=-\sin x\,dx\end{matrix}} = -\int \frac{dt}{1-t^2} = \int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c = \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + c =$$

$$= \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + c = \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + c, \qquad \text{for } x\in R-\{k\pi\,;\,k\in Z\}.$$

$$\int \frac{dx}{1+\sin x} = \boxed{t=\operatorname{tg}\frac{x}{2},\ dx=\frac{2\,dt}{1+t^2},\ 1+\sin x=1+\frac{2t}{1+t^2}=\frac{1+2t+t^2}{1+t^2}} = \int \frac{1+t^2}{(1+t)^2}\frac{2\,dt}{1+t^2} = \int \frac{2\,dt}{(1+t)^2} = \boxed{\begin{matrix}1+t=u\\ dt=du\end{matrix}} =$$

$$= 2\int \frac{du}{u^2} = 2\int u^{-2}\,du = 2\frac{u^{-1}}{-1} + c = c - \frac{2}{u} = c - \frac{2}{t+1} = c - \frac{2}{\operatorname{tg}\frac{x}{2}+1}, \qquad \text{for } x\in R-\left\{\frac{\pi}{2}+2k\pi\,;\,k\in Z\right\}.$$

$$\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}\,dx = \int \frac{1-\sin x}{1-\sin^2 x}\,dx = \int \frac{dx}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x}\,dx = \boxed{\begin{matrix}\cos x=t\\ -\sin x\,dx=dt\end{matrix}} = \operatorname{tg} x + \int \frac{dt}{t^2} =$$

$$= \operatorname{tg} x + \frac{t^{-1}}{-1} + c = \operatorname{tg} x - \frac{1}{t} + c = \operatorname{tg} x - \frac{1}{\cos x} + c = \frac{\sin x}{\cos x} - \frac{1}{\cos x} + c = \frac{\sin x-1}{\cos x} + c, \qquad \text{for } x\in R-\left\{\frac{\pi}{2}+k\pi\,;\,k\in Z\right\}.$$

$$\int \frac{\sin x\,\mathrm{d}x}{\sqrt{\cos^5 x}} = \boxed{\begin{array}{c}\cos x = t\\ -\sin x\,\mathrm{d}x = \mathrm{d}t\end{array}} = \int \frac{-\,\mathrm{d}t}{\sqrt{t^5}} = -\int t^{-\frac{5}{2}}\,\mathrm{d}t = -\frac{t^{-\frac{3}{2}}}{-\frac{3}{2}} + c = \frac{2}{3}\frac{1}{\sqrt{t^3}} + c = \frac{2}{3}\frac{1}{\sqrt{\cos^3 x}} + c, \quad \text{for } x \in \bigcup_{k\in Z}\left(-\frac{\pi}{2} + 2k\pi\,;\,\frac{\pi}{2} + 2k\pi\right).$$

$$\int \frac{\cos x\,\mathrm{d}x}{\sqrt[3]{\sin^2 x}} = \boxed{\begin{array}{c}\sin x = t\\ \cos x\,\mathrm{d}x = \mathrm{d}t\end{array}} = \int \frac{\mathrm{d}t}{\sqrt[3]{t^2}} = \int t^{-\frac{2}{3}}\,\mathrm{d}t = \frac{t^{\frac{1}{3}}}{\frac{1}{3}} + c = 3\sqrt[3]{t} + c = 3\sqrt[3]{\sin x} + c, \qquad \text{for } x \in \bigcup_{k\in Z}(0 + 2k\pi\,;\,\pi + 2k\pi).$$

$$\int \sin^2 x\,\mathrm{d}x = \int \frac{1-\cos 2x}{2}\,\mathrm{d}x = \frac{x}{2} - \frac{\sin 2x}{4} + c, \qquad \text{for } x \in R.$$

$$\int \cos^2 x\,\mathrm{d}x = \int \frac{1+\cos 2x}{2}\,\mathrm{d}x = \frac{x}{2} + \frac{\sin 2x}{4} + c, \qquad \text{for } x \in R.$$

$$\int \sin^3 x\,\mathrm{d}x = \int (1-\cos^2 x)\sin x\,\mathrm{d}x = \boxed{\begin{array}{c}\cos x = t\\ -\sin x\,\mathrm{d}x = \mathrm{d}t\end{array}} = -\int (1-t^2)\,\mathrm{d}t = -t + \frac{t^3}{3} + c = \frac{\cos^3 x}{3} - \cos x + c, \qquad \text{for } x \in R.$$

$$\int \cos^3 x\,\mathrm{d}x = \int (1-\sin^2 x)\sin x\,\mathrm{d}x = \boxed{\begin{array}{c}\sin x = t\\ \cos x\,\mathrm{d}x = \mathrm{d}t\end{array}} = \int (1-t^2)\,\mathrm{d}t = t - \frac{t^3}{3} + c = \sin x - \frac{\sin^3 x}{3} + c, \qquad \text{for } x \in R.$$

$$\int \sin^n x\,\mathrm{d}x = \int \sin x \sin^{n-1} x\,\mathrm{d}x = \boxed{\begin{array}{lcl}u = \sin^{n-1} x & \Rightarrow & u' = (n-1)\sin^{n-2} x\cos x\\ v' = \sin x & \Rightarrow & v = -\cos x\end{array}} = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\cos^2 x\,\mathrm{d}x =$$

$$= -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x(1-\sin^2 x)\,\mathrm{d}x = -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\,\mathrm{d}x - (n-1)\int \sin^n x\,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} \; -\cos x\sin^{n-1} x + (n-1)\int \sin^{n-2} x\,\mathrm{d}x = \int \sin^n x\,\mathrm{d}x + (n-1)\int \sin^n x\,\mathrm{d}x = n\int \sin^n x\,\mathrm{d}x$$

$$\Longrightarrow \int \sin^n x\,\mathrm{d}x = -\frac{\cos x\sin^{n-1} x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\,\mathrm{d}x, \qquad \text{for } n = 3, 4, 5, \ldots,\ x \in R.$$

$$\int \sin^{2k+1} x\,\mathrm{d}x = \int \sin x\sin^{2k} x\,\mathrm{d}x = \int \sin x(\sin^2 x)^k\,\mathrm{d}x = \int \sin x(1-\cos^2 x)^k\,\mathrm{d}x = \boxed{\begin{array}{c}\cos x = t\\ -\sin x\,\mathrm{d}x = \mathrm{d}t\end{array}} = -\int (1-t^2)^k\,\mathrm{d}t =$$

$$= -\int \left[\sum_{j=0}^{k}\binom{k}{j}(-t^2)^j\right]\mathrm{d}t = -\int \left[\sum_{j=0}^{k}\binom{k}{j}(-1)^j t^{2j}\right]\mathrm{d}t = \sum_{j=0}^{k}\binom{k}{j}(-1)^{j+1}\frac{t^{2j+1}}{2j+1} + c = \sum_{j=0}^{k}\binom{k}{j}(-1)^{j+1}\frac{\cos^{2j+1} x}{2j+1} + c =$$

$$= -\cos x + \binom{k}{1}\frac{\cos^3 x}{3} - \cdots + \binom{k}{k-1}(-1)^k\frac{\cos^{2k-1} x}{2k-1} + (-1)^{k+1}\frac{\cos^{2k+1} x}{2k+1} + c, \qquad \text{for } k \in N,\ x \in R.$$

$$\int \mathrm{tg}^2 x\,\mathrm{d}x = \int \frac{\sin^2 x}{\cos^2 x}\,\mathrm{d}x = \int \frac{1-\cos^2 x}{\cos^2 x}\,\mathrm{d}x = \int \left[\frac{1}{\cos^2 x} - 1\right]\mathrm{d}x = \mathrm{tg}\,x - x + c, \qquad \text{for } x \in R - \left\{\frac{\pi}{2} + k\pi\,;\,k \in Z\right\}.$$

$$\int \cos^n x\,\mathrm{d}x = \int \cos x \cos^{n-1} x\,\mathrm{d}x = \boxed{\begin{array}{ll} u=\cos^{n-1} x & \Rightarrow\ u'=-(n-1)\cos^{n-2} x \sin x \\ v'=\cos x & \Rightarrow\ v=\sin x \end{array}} = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x\,\mathrm{d}x =$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,\mathrm{d}x = \sin x\cos^{n-1} x + (n-1)\int \cos^{n-2} x\,\mathrm{d}x - (n-1)\int \cos^n x\,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x\,\mathrm{d}x = \int \cos^n x\,\mathrm{d}x + (n-1)\int \cos^n x\,\mathrm{d}x = n\int \cos^n x\,\mathrm{d}x$$

$$\Longrightarrow \int \cos^n x\,\mathrm{d}x = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,\mathrm{d}x, \qquad \text{for}\ \ n=3,4,5,\ldots,\ x\in R.$$

$$\int \cos^{2k+1} x\,\mathrm{d}x = \int \cos x \cos^{2k} x\,\mathrm{d}x = \int \cos x(\cos^2 x)^k\,\mathrm{d}x = \int \cos x(1-\sin^2 x)^k\,\mathrm{d}x = \boxed{\begin{array}{c} \sin x = t \\ \cos x\,\mathrm{d}x = \mathrm{d}t \end{array}} = \int (1-t^2)^k\,\mathrm{d}t =$$

$$= \int \left[\sum_{j=0}^{k} \binom{k}{j} (-t^2)^j\right]\mathrm{d}t = \int \left[\sum_{j=0}^{k} \binom{k}{j} (-1)^j t^{2j}\right]\mathrm{d}t = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{t^{2j+1}}{2j+1} + c = \sum_{j=0}^{k} \binom{k}{j} (-1)^j \frac{\sin^{2j+1} x}{2j+1} + c =$$

$$= \sin x - \binom{k}{1}\frac{\sin^3 x}{3} - \cdots + \binom{k}{k-1}(-1)^{k-1}\frac{\sin^{2k-1} x}{2k-1} + (-1)^k \frac{\sin^{2k+1} x}{2k+1} + c, \qquad \text{for}\ \ k\in N,\ x\in R.$$

$$\int \frac{\sin x - \cos x}{\sqrt[4]{\sin x + \cos x}}\,\mathrm{d}x = \boxed{\begin{array}{c} t = \operatorname{tg}\frac{x}{2},\ \mathrm{d}x = \frac{2\,\mathrm{d}t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2},\ \sin x = \frac{2t}{1+t^2} \end{array}} = \int \frac{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2}}{\sqrt[4]{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}}}\,\frac{2\,\mathrm{d}t}{1+t^2} = \int (t^2+2t-1)\sqrt[4]{\frac{1+t^2}{1+2t-t^2}}\,\mathrm{d}t = \cdots. \quad ☹$$

$$\int \frac{\sin x - \cos x}{\sqrt[4]{\sin x + \cos x}}\,\mathrm{d}x = \boxed{\begin{array}{c} \sin x + \cos x = t \\ (\cos x - \sin x)\,\mathrm{d}x = \mathrm{d}t \end{array}} = \int \frac{-\mathrm{d}t}{\sqrt[4]{t}} = -\int t^{-\frac{1}{4}}\,\mathrm{d}t = -\frac{t^{\frac{3}{4}}}{\frac{3}{4}} + c = c - \frac{4}{3}\sqrt[4]{(\sin x + \cos x)^3},$$

$$\text{for}\ \ x\in \bigcup_{k\in Z}\left(\frac{\pi}{2} + 2k\pi\,;\ \frac{3\pi}{2} + 2k\pi\right).$$

$$\int \frac{x^2}{\sin x^3}\,\mathrm{d}x = \boxed{\begin{array}{c} t = x^3 \\ \mathrm{d}t = 3x^2\,\mathrm{d}x \end{array}} = \frac{1}{3}\int \frac{\mathrm{d}t}{\sin t} = \boxed{u = \operatorname{tg}\frac{t}{2},\ \mathrm{d}t = \frac{2\,\mathrm{d}u}{1+u^2},\ \sin t = \frac{2u}{1+u^2}} = \frac{1}{3}\int \frac{1+u^2}{2u}\,\frac{2\,\mathrm{d}u}{1+u^2} = \frac{1}{3}\int \frac{\mathrm{d}u}{u} =$$

$$= \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln\left|\operatorname{tg}\frac{t}{2}\right| + c = \ln\sqrt[3]{\left|\operatorname{tg}\frac{x^3}{2}\right|} + c, \qquad \text{for}\ \ x\in R - \left\{\sqrt[3]{k\pi}\,;\ k\in Z\right\}.$$

$$\int \frac{\mathrm{d}x}{\sin^2 x \cos^2 x} = \int \frac{4\,\mathrm{d}x}{(2\sin x\cos x)^2} = \int \frac{4\,\mathrm{d}x}{\sin^2 2x} = \boxed{\begin{array}{c} 2x = d \\ 2\,\mathrm{d}x = \mathrm{d}t \end{array}} = \int \frac{2\,\mathrm{d}t}{\sin^2 t} = -2\operatorname{cotg} t + c = -2\operatorname{cotg} 2x + c, \ \text{for}\ x\in R - \left\{\frac{k\pi}{2}\,;\ k\in Z\right\}.$$

$$\int \frac{\mathrm{d}x}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\,\mathrm{d}x = \int \frac{\mathrm{d}x}{\sin^2 x} + \int \frac{\mathrm{d}x}{\cos^2 x} = \operatorname{tg} x - \operatorname{cotg} x + c =$$

$$= \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} + c = 2\frac{\sin^2 x - \cos^2 x}{2\sin x\cos x} + c = 2\frac{-\cos 2x}{\sin 2x} + c = -2\operatorname{cotg} 2x + c, \ \text{for}\ x\in R - \left\{\frac{k\pi}{2}\,;\ k\in Z\right\}.$$

$$\int \cos ax \sin bx \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\sin bx \ \Rightarrow\ u'=b\cos bx \\ v'=\cos ax \ \Rightarrow\ v=\dfrac{\sin ax}{a}\end{array}} = \frac{\sin ax \sin bx}{a} - \frac{b}{a}\int \sin ax \cos bx \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\cos bx \ \Rightarrow\ u'=-b\sin bx \\ v'=\sin ax \ \Rightarrow\ v=-\dfrac{\cos ax}{a}\end{array}} =$$

$$= \frac{\sin ax \sin bx}{a} - \frac{b}{a}\left[-\frac{\cos ax \cos bx}{a} - \frac{b}{a}\int \cos ax \sin bx \,\mathrm{d}x\right] = \frac{\sin ax \sin bx}{a} + \frac{b}{a^2}\cos ax \cos bx + \frac{b^2}{a^2}\int \cos ax \sin bx \,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} \frac{\sin ax \sin bx}{a} + \frac{b}{a^2}\cos ax \cos bx + c = \int \cos ax \sin bx \,\mathrm{d}x - \frac{b^2}{a^2}\int \cos ax \sin bx \,\mathrm{d}x = \frac{a^2-b^2}{a^2}\int \cos ax \sin bx \,\mathrm{d}x$$

$$\Longrightarrow \int \cos ax \sin bx \,\mathrm{d}x = \frac{a}{a^2-b^2}\sin ax \sin bx + \frac{b}{a^2-b^2}\cos ax \cos bx + c, \qquad \text{for } a,b\in R-\{0\},\ a\neq b,\ x\in R.$$

$$\int \cos ax \sin bx \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\cos ax \ \Rightarrow\ u'=-a\sin ax \\ v'=\sin bx \ \Rightarrow\ v=-\dfrac{\cos bx}{b}\end{array}} = -\frac{\cos ax \cos bx}{b} - \frac{a}{b}\int \sin ax \cos bx \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\sin ax \ \Rightarrow\ u'=a\cos ax \\ v'=\cos bx \ \Rightarrow\ v=\dfrac{\sin bx}{b}\end{array}} =$$

$$= -\frac{\cos ax \cos bx}{b} - \frac{a}{b}\left[\frac{\sin ax \sin bx}{b} - \frac{a}{b}\int \cos ax \sin bx \,\mathrm{d}x\right] = -\frac{\cos ax \cos bx}{b} - \frac{a}{b^2}\sin ax \sin bx + \frac{a^2}{b^2}\int \cos ax \sin bx \,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} c - \frac{\cos ax \cos bx}{b} - \frac{a}{b^2}\sin ax \sin bx = \int \cos ax \sin bx \,\mathrm{d}x - \frac{a^2}{b^2}\int \cos ax \sin bx \,\mathrm{d}x = \frac{b^2-a^2}{b^2}\int \cos ax \sin bx \,\mathrm{d}x$$

$$\Longrightarrow \int \cos ax \sin bx \,\mathrm{d}x = \frac{b}{a^2-b^2}\cos ax \cos bx + \frac{a}{a^2-b^2}\sin ax \sin bx + c, \qquad \text{for } a,b\in R-\{0\},\ a\neq b,\ x\in R.$$

$$\int \cos ax \sin ax \,\mathrm{d}x = \int \frac{\sin 2ax}{2}\,\mathrm{d}x = \frac{-\cos 2ax}{2\cdot 2a} + c = c - \frac{\cos 2ax}{4a}, \qquad \text{for } a=b,\ a\in R-\{0\},\ x\in R.$$

$$\int x\ln^2 x \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\ln^2 x \ \Rightarrow\ u'=\dfrac{2\ln x}{x} \\ v'=x \ \Rightarrow\ v=\dfrac{x^2}{2}\end{array}} = \frac{x^2}{2}\ln^2 x - \int \frac{x^2}{2}\frac{2\ln x}{x}\,\mathrm{d}x = \frac{x^2}{2}\ln^2 x - \int x\ln x \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\ln x \ \Rightarrow\ u'=\dfrac{1}{x} \\ v'=x \ \Rightarrow\ v=\dfrac{x^2}{2}\end{array}} =$$

$$= \frac{x^2}{2}\ln^2 x - \left[\frac{x^2}{2}\ln x - \int \frac{x^2}{2}\frac{1}{x}\,\mathrm{d}x\right] = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \int \frac{x\,\mathrm{d}x}{2} = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + c, \qquad \text{for } x>0.$$

$$\int \frac{\mathrm{d}x}{(1-x)x^2} = \int \left[\frac{1}{x-1} - \frac{1}{x^2} - \frac{1}{x}\right]\mathrm{d}x = \ln|x-1| - \frac{x^{-1}}{-1} - \ln|x| + c = \ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + c, \qquad \text{for } x\in R-\{0,1\}.$$

$$\boxed{\frac{1}{(1-x)x^2} = \frac{A}{x-1} + \frac{B}{x^2} + \frac{C}{x} \Longrightarrow 1 = Ax^2 + B(x-1) + Cx(x-1) \Longrightarrow A=1,\ B=-1,\ C=-1}$$

$$\int \ln x \,\mathrm{d}x = \boxed{\begin{array}{ll} u=\ln x \ \Rightarrow\ u'=\dfrac{1}{x} \\ v'=1 \ \Rightarrow\ v=x\end{array}} = x\ln x - \int x\frac{\mathrm{d}x}{x} = x\ln x - \int \mathrm{d}x = x\ln x - x + c, \qquad \text{for } x>0.$$

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \boxed{\begin{array}{c} x = a\sin t \\ \mathrm{d}x = a\cos t\,\mathrm{d}t \end{array}} = \int a^2\cos^2 t\,\mathrm{d}t = a^2\int \frac{1+\cos 2t}{2}\,\mathrm{d}t = \frac{a^2}{2}t + a^2\frac{\sin 2t}{4} + c_1 = \frac{a^2}{2}t + a^2\frac{2\sin t\cos t}{4} + c_1 =$$

$$\boxed{t = \arcsin\frac{x}{a}, \quad \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 t} = |a|\sqrt{1-\sin^2 t} = a\sqrt{\cos^2 t} = a\,|\cos t| = a\cos t, \quad x\in(-a\,;\,a) \;\Rightarrow\; t\in\left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right)}$$

$$= \frac{a^2}{2}t + \frac{a\sin t\cdot a\cos t}{2} + c_1 = \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2} + c_1, \qquad \text{for } a>0,\ x\in(-a\,;\,a).$$

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \int \sqrt{(-a)^2-x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\frac{x}{-a} + \frac{x\sqrt{a^2-x^2}}{2} + c_1, \qquad \text{for } a<0 \text{ [i.e. } -a>0,\ (-a)^2=a^2],\ x\in(a\,;\,-a).$$

$$\Longrightarrow \int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\frac{x}{|a|} + \frac{x\sqrt{a^2-x^2}}{2} + c_1, \qquad \text{for } a\in R-\{0\},\ x\in(-|a|\,;\,|a|).$$

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \int \frac{a^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} - \int \frac{x^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} = \boxed{\begin{array}{ll} u = x & \Rightarrow\ u' = 1 \\ v' = \dfrac{x}{\sqrt{a^2-x^2}} & \Rightarrow\ v = -\sqrt{a^2-x^2} \end{array}} =$$

$$= \int \frac{a^2\,\mathrm{d}x}{\sqrt{a^2-x^2}} - \left[-x\sqrt{a^2-x^2} + \int \sqrt{a^2-x^2}\,\mathrm{d}x\right] = a^2\arcsin\frac{x}{|a|} + x\sqrt{a^2-x^2} - \int \sqrt{a^2-x^2}\,\mathrm{d}x$$

$$\xRightarrow{\text{(i.e.equation)}} a^2\arcsin\frac{x}{|a|} + x\sqrt{a^2-x^2} + c_0 = \int \sqrt{a^2-x^2}\,\mathrm{d}x + \int \sqrt{a^2-x^2}\,\mathrm{d}x = 2\int \sqrt{a^2-x^2}\,\mathrm{d}x$$

$$\Longrightarrow \int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{a^2}{2}\arcsin\frac{x}{|a|} + \frac{x\sqrt{a^2-x^2}}{2} + c_1, \qquad \text{for } a\in R-\{0\},\ x\in(-|a|\,;\,|a|).$$

$$\int \sqrt{a^2-x^2}\,\mathrm{d}x = \int \frac{a(1-t^2)}{1+t^2}\frac{2a(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t = 2a^2\int \frac{(1-t^2)^2}{(1+t^2)^3}\,\mathrm{d}t = 2a^2\int \frac{1-2t^2+t^4}{(1+t^2)^3}\,\mathrm{d}t = 2a^2\int \left[\frac{1+2t^2+t^4-4t^2}{(1+t^2)^3}\right]\mathrm{d}t =$$

$$\boxed{\begin{array}{l} \text{2-nd Euler:} \quad \sqrt{a^2-x^2} = a-xt \;\Rightarrow\; a^2-x^2 = x^2t^2-2axt+a^2 \;\Rightarrow\; 2axt = x^2+x^2t^2 \;\Rightarrow\; x=0 \text{ or } 2at = x+xt^2 \\ x = \dfrac{2at}{1+t^2}, \quad \mathrm{d}x = \dfrac{2a(1+t^2)-2at\cdot 2t}{(1+t^2)^2}\,\mathrm{d}t = \dfrac{2a(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t, \quad a-xt = a-\dfrac{2at^2}{1+t^2} = \dfrac{a+at^2-2at^2}{1+t^2} = \dfrac{a(1-t^2)}{1+t^2} \end{array}}$$

$$= 2a^2\int \left[\frac{(1+t^2)^2-4t^2-4+4}{(1+t^2)^3}\right]\mathrm{d}t = 2a^2\int \left[\frac{(1+t^2)^2-4(1+t^2)+4}{(1+t^2)^3}\right]\mathrm{d}t = 2a^2\int \left[\frac{1}{1+t^2} - \frac{4}{(1+t^2)^2} + \frac{4}{(1+t^2)^3}\right]\mathrm{d}t =$$

$$= 2a^2\left[\operatorname{arctg} t - 4\left(\frac{1}{2}\operatorname{arctg} t + \frac{1}{2}\frac{t}{1+t^2}\right) + 4\left(\frac{3}{8}\operatorname{arctg} t + \frac{3}{8}\frac{t}{1+t^2} + \frac{1}{4}\frac{t}{(1+t^2)^2}\right)\right] + c_2 = a^2\operatorname{arctg} t - \frac{a^2t}{1+t^2} + \frac{2a^2t}{(1+t^2)^2} + c_2 =$$

$$= \boxed{x = \frac{2at}{1+t^2}, \quad \sqrt{a^2-x^2} = a-xt \;\Rightarrow\; t = \frac{a-\sqrt{a^2-x^2}}{x}} = a^2\operatorname{arctg} t - \frac{a\cdot 2at}{2(1+t^2)} + \frac{4a^2t^2}{2t(1+t^2)^2} + c_2 =$$

$$= a^2\operatorname{arctg}\frac{a-\sqrt{a^2-x^2}}{x} - \frac{ax}{2} + \frac{x^3}{2\left(a-\sqrt{a^2-x^2}\right)} + c_2 = a^2\operatorname{arctg}\frac{a-\sqrt{a^2-x^2}}{x} - \frac{ax}{2} + \frac{x^3}{2\left(a-\sqrt{a^2-x^2}\right)}\frac{a+\sqrt{a^2-x^2}}{a+\sqrt{a^2-x^2}} + c_2 =$$

$$= a^2\operatorname{arctg}\frac{a-\sqrt{a^2-x^2}}{x} - \frac{ax}{2} + \frac{x}{2}\left(a+\sqrt{a^2-x^2}\right) + c_2 = a^2\operatorname{arctg}\frac{a-\sqrt{a^2-x^2}}{x} + \frac{x\sqrt{a^2-x^2}}{2} + c_2,$$

$$\text{for } a\in R-\{0\},\ x\in(-|a|\,;\,|a|).$$

$$\int \sqrt{a^2+x^2}\,\mathrm{d}x = \boxed{\begin{array}{c} x = a\sinh t \\ \mathrm{d}x = a\cosh t\,\mathrm{d}t \end{array}} = \int a^2\cosh^2 t\,\mathrm{d}t = a^2\int \frac{(\mathrm{e}^t+\mathrm{e}^{-t})^2}{4}\,\mathrm{d}t = a^2\int \frac{\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}}{4}\,\mathrm{d}t = a^2\left[\frac{\mathrm{e}^{2t}}{8}+\frac{t}{2}-\frac{\mathrm{e}^{-2t}}{8}\right]+c=$$

$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\sinh^2 t} = |a|\,\sqrt{1+\sinh^2 t} = a\sqrt{\cosh^2 t} = a\,|\cosh t| = a\cosh t,\ \ t\in R$$

Let $u=:\ \mathrm{e}^t>0 \implies x = a\sinh t = \frac{a}{2}(\mathrm{e}^t-\mathrm{e}^{-1}) = \frac{a}{2}(u-u^{-1}) \implies \frac{2x}{a} = u-u^{-1} \implies u^2-\frac{2x}{a}u-1=0 \implies$

$$u_{1,2} = \frac{\frac{2x}{a}\pm\sqrt{\frac{4x^2}{a^2}+4}}{2} = \frac{\frac{2x}{a}\pm\sqrt{\frac{4x^2+4a^2}{a^2}}}{2} = \frac{\frac{2x}{a}\pm\frac{2}{a}\sqrt{a^2+x^2}}{2} = \frac{x\pm\sqrt{a^2+x^2}}{a} \implies \mathrm{e}^t = \frac{x+\sqrt{a^2+x^2}}{a} \implies$$

$$\mathrm{e}^{-2t} = \frac{a^2}{\left(x+\sqrt{a^2+x^2}\right)^2} = \frac{a^2}{\left(x+\sqrt{a^2+x^2}\right)^2}\,\frac{\left(x-\sqrt{a^2+x^2}\right)^2}{\left(x-\sqrt{a^2+x^2}\right)^2} = \frac{a^2\left(x-\sqrt{a^2+x^2}\right)^2}{[x^2-(a^2+x^2)]^2} = \frac{\left(x-\sqrt{a^2+x^2}\right)^2}{a^2}$$

$$\sqrt{a^2+x^2} > \sqrt{x^2} \geq x \implies \sqrt{a^2+x^2}-x>0 \implies t = \ln\frac{x+\sqrt{a^2+x^2}}{a} = \ln\left(x+\sqrt{a^2+x^2}\right)-\ln a$$

$$= \frac{\left(x+\sqrt{a^2+x^2}\right)^2}{8} + a^2\frac{\ln\left(x+\sqrt{a^2+x^2}\right)-\ln a}{2} - \frac{\left(x-\sqrt{a^2+x^2}\right)^2}{8}+c_1 =$$

$$= \frac{x^2+2x\sqrt{a^2+x^2}+a^2+x^2}{8} - \frac{x^2-2x\sqrt{a^2+x^2}+a^2+x^2}{8} + a^2\frac{\ln\left(x+\sqrt{a^2+x^2}\right)}{2} - \frac{a^2}{2}\ln a + c_1 =$$

$$= \frac{4x\sqrt{a^2+x^2}}{8} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+\ c, \qquad \text{for}\ \ a>0,\ x\in R.$$

$$\int \sqrt{a^2+x^2}\,\mathrm{d}x = \int \sqrt{(-a)^2+x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+\ c, \qquad \text{for}\ \ a<0\ [\text{i.e. } -a>0,\ (-a)^2=a^2],\ x\in R.$$

$$\implies \int \sqrt{a^2+x^2}\,\mathrm{d}x = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+\ c, \qquad \text{for}\ \ a\in R-\{0\},\ x\in R.$$

$$\int \sqrt{a^2+x^2}\,\mathrm{d}x = \int \frac{t^2+a^2}{2t}\,\frac{t^2+a^2}{2t^2}\,\mathrm{d}t = \int \frac{t^4+2t^2a^2+a^4}{4t^3}\,\mathrm{d}t = \frac{1}{4}\int \left[t+2a^2t^{-1}+a^4t^{-3}\right]\mathrm{d}t = \frac{1}{4}\left[\frac{t^2}{2}+2a^2\ln|t|+a^4\frac{t^{-2}}{-2}\right]+c=$$

1-st Euler: $\sqrt{a^2+x^2} = t-x \implies a^2+x^2 = t^2-2tx+x^2 \implies x = \frac{t^2-a^2}{2t}, \quad t = x+\sqrt{a^2+x^2}$

$$\sqrt{a^2+x^2} = t-\frac{t^2-a^2}{2t} = \frac{t^2+a^2}{2t}, \qquad \mathrm{d}x = \frac{2t\cdot 2t-2(t^2-a^2)}{4t^2}\,\mathrm{d}t = \frac{2t^2+2a^2}{4t^2}\,\mathrm{d}t = \frac{t^2+a^2}{2t^2}\,\mathrm{d}t$$

$$t^2 = \left(x+\sqrt{a^2+x^2}\right)^2, \quad t^{-2} = \frac{1}{\left(x+\sqrt{a^2+x^2}\right)^2}\,\frac{\left(x-\sqrt{a^2+x^2}\right)^2}{\left(x-\sqrt{a^2+x^2}\right)^2} = \frac{\left(x-\sqrt{a^2+x^2}\right)^2}{[x^2-(a^2+x^2)]^2} = \frac{\left(x-\sqrt{a^2+x^2}\right)^2}{a^4}$$

$$= \frac{\left(x+\sqrt{a^2+x^2}\right)^2}{8} - \frac{\left(x-\sqrt{a^2+x^2}\right)^2}{8} + \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right|+c=$$

$$= \frac{x^2+2x\sqrt{a^2+x^2}+a^2+x^2}{8} - \frac{x^2-2x\sqrt{a^2+x^2}+a^2+x^2}{8} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c=$$

$$= \frac{4x\sqrt{a^2+x^2}}{8} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+c = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right)+\ c, \qquad \text{for}\ \ a\in R-\{0\},\ x\in R.$$

$$\int \sqrt{a^2+x^2}\,dx = \int \frac{a^2+x^2}{\sqrt{a^2+x^2}}\,dx = \int \frac{a^2\,dx}{\sqrt{a^2+x^2}} + \int \frac{x^2\,dx}{\sqrt{a^2+x^2}} = \boxed{\begin{array}{ll} u=x & \Rightarrow\ u'=1 \\ v'=\dfrac{x}{\sqrt{a^2+x^2}} & \Rightarrow\ v=\sqrt{a^2+x^2} \end{array}} =$$

$$= \int \frac{a^2\,dx}{\sqrt{a^2+x^2}} + \left[x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx\right] = a^2 \ln\left(x+\sqrt{a^2+x^2}\right) + x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx$$

$$\xRightarrow{\text{(i.e.equation)}} a^2 \ln\left(x+\sqrt{a^2+x^2}\right) + x\sqrt{a^2+x^2} + c_0 = \int \sqrt{a^2+x^2}\,dx + \int \sqrt{a^2+x^2}\,dx = 2\int \sqrt{a^2+x^2}\,dx$$

$$\Longrightarrow \int \sqrt{a^2+x^2}\,dx = \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + \frac{x\sqrt{a^2+x^2}}{2} + c, \qquad \text{for } a\in R-\{0\},\ x\in R.$$

$$\int \sqrt{a^2+x^2}\,dx = \int \frac{a^2\,dt}{\cos^3 t} = \int \frac{a^2 \cos t\,dt}{\cos^4 t} = \int \frac{a^2 \cos t\,dt}{\left(1-\sin^2 t\right)^2} = \boxed{\begin{array}{c} u=\sin t \\ du=\cos t\,dt \end{array}} = \int \frac{a^2\,du}{(1-u^2)^2} = \int \frac{a^2\,du}{(u^2-1)^2} =$$

$$\boxed{\begin{array}{c} x=a\,\mathrm{tg}\,t \implies dx=\dfrac{a\,dt}{\cos^2 t} \implies \sqrt{a^2+x^2}=\sqrt{a^2+\dfrac{a^2\sin^2 t}{\cos^2 t}}=\sqrt{\dfrac{a^2\cos^2 t+a^2\sin^2 t}{\cos^2 t}}=\sqrt{\dfrac{a^2}{\cos^2 t}}=\dfrac{a}{\cos t} \\ x\in R \implies t=\mathrm{arctg}\,x\in\left(-\dfrac{\pi}{2};\dfrac{\pi}{2}\right) \implies \sqrt{\cos^2 x}=|\cos x|=\cos x,\quad \mathrm{tg}^2 t=\dfrac{\sin^2 t}{\cos^2 t}=\dfrac{\sin^2 t}{1-\sin^2 t} \implies \sin^2 t=\dfrac{\mathrm{tg}^2 t}{1+\mathrm{tg}^2 t} \\ \mathrm{sgn}\sin t=\mathrm{sgn}\,\mathrm{tg}\,t \implies \sin t=\dfrac{\mathrm{tg}\,t}{\sqrt{1+\mathrm{tg}^2 t}}=\dfrac{\frac{x}{a}}{\sqrt{1+\frac{x^2}{a^2}}}=\dfrac{x}{\sqrt{a^2+x^2}} \implies \sin t\pm 1=\dfrac{x\pm\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}} \end{array}}$$

$$= \int \frac{a^2\,du}{(u-1)^2(u+1)^2} = \frac{a^2}{4}\int\left[\frac{1}{u+1}-\frac{1}{u-1}+\frac{1}{(u+1)^2}+\frac{1}{(u-1)^2}\right]du = \boxed{\begin{array}{c} u-1=r \\ du=dr \end{array}}\ \boxed{\begin{array}{c} u+1=s \\ du=ds \end{array}} =$$

$$\boxed{\frac{1}{(u-1)^2(u+1)^2}=\frac{A}{u+1}+\frac{B}{(u+1)^2}+\frac{C}{u-1}+\frac{D}{(u-1)^2} \implies A=\frac{1}{4},\ B=\frac{1}{4},\ C=-\frac{1}{4},\ D=\frac{1}{4}}$$

$$= \frac{a^2}{4}\left[\ln|u+1|-\ln|u-1|+\int r^{-2}\,dr+\int s^{-2}\,ds\right] = \frac{a^2}{4}\ln\left|\frac{u+1}{u-1}\right| + \frac{a^2}{4}\left[\frac{r^{-1}}{-1}+\frac{s^{-1}}{-1}\right] + c_1 =$$

$$= \frac{a^2}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{a^2}{4}\left[\frac{1}{u-1}+\frac{1}{u+1}\right] + c_1 = \frac{a^2}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{2ua^2}{4(u^2-1)} + c_1 = \frac{a^2}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{ua^2}{2(u^2-1)} + c_1 =$$

$$= \frac{a^2}{4}\ln\left|\frac{\sin t+1}{\sin t-1}\right| - \frac{a^2\sin t}{2(\sin^2 t-1)} + c_1 = \frac{a^2}{4}\ln\left|\frac{x+\sqrt{a^2+x^2}}{x-\sqrt{a^2+x^2}}\right| - \frac{xa^2}{2\sqrt{a^2+x^2}}\left[\frac{x^2}{a^2+x^2}-1\right]^{-1} + c_1 =$$

$$= \frac{a^2}{4}\ln\left|\frac{x+\sqrt{a^2+x^2}}{x-\sqrt{a^2+x^2}}\cdot\frac{x+\sqrt{a^2+x^2}}{x+\sqrt{a^2+x^2}}\right| - \frac{xa^2}{2\sqrt{a^2+x^2}}\,\frac{a^2+x^2}{-a^2} + c_1 = \frac{a^2}{4}\ln\left|\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{x^2-(a^2+x^2)}\right| + \frac{x\sqrt{a^2+x^2}}{2} + c_1 =$$

$$= \frac{a^2}{4}\ln\left|\frac{\left(x+\sqrt{a^2+x^2}\right)^2}{a^2}\right| + \frac{x\sqrt{a^2+x^2}}{2} + c_1 = \frac{a^2}{2}\ln\left|x+\sqrt{a^2+x^2}\right| - \frac{a^2}{2}\ln a + \frac{x\sqrt{a^2+x^2}}{2} + c_1 =$$

$$= \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + \frac{x\sqrt{a^2+x^2}}{2} + c, \qquad \text{for } a>0,\ x\in R.$$

$$\int \sqrt{a^2+x^2}\,dx = \int \sqrt{(-a)^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + c, \qquad \text{for } a<0 \text{ [i.e. } -a>0,\ (-a)^2=a^2],\ x\in R.$$

$$\Longrightarrow \int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\ln\left(x+\sqrt{a^2+x^2}\right) + c, \qquad \text{for } a\in R-\{0\},\ x\in R.$$

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \boxed{\begin{matrix} x = a\cosh t \\ \mathrm{d}x = a\sinh t\,\mathrm{d}t \end{matrix}} = \int a^2\sinh^2 t\,\mathrm{d}t = a^2\int \frac{(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}\,\mathrm{d}t = a^2\int \frac{\mathrm{e}^{2t}-2+\mathrm{e}^{-2t}}{4}\,\mathrm{d}t = a^2\left[\frac{\mathrm{e}^{2t}}{8}-\frac{t}{2}-\frac{\mathrm{e}^{-2t}}{8}\right]+c=$$

$$\sqrt{x^2-a^2} = \sqrt{a^2\cosh^2 t - a^2} = a\sqrt{\cosh^2 t - 1} = a\sqrt{\sinh^2 t} = a\,|\sinh t| = a\sinh t, \quad t>0, \quad x>a$$

$$u =: \mathrm{e}^t > 0 \qquad \Longrightarrow\ x = a\cosh t = \frac{a}{2}(\mathrm{e}^t+\mathrm{e}^{-1}) = \frac{a}{2}(u+u^{-1}) \ \Longrightarrow\ \frac{2x}{a} = u+u^{-1} \ \Longrightarrow\ u^2 - \frac{2x}{a}u + 1 = 0 \ \Longrightarrow$$

$$u_{1,2} = \frac{\frac{2x}{a} \pm \sqrt{\frac{4x^2}{a^2}-4}}{2} = \frac{\frac{2x}{a} \pm \sqrt{\frac{4x^2-4a^2}{a^2}}}{2} = \frac{\frac{2x}{a} \pm \frac{2}{a}\sqrt{x^2-a^2}}{2} = \frac{x \pm \sqrt{x^2-a^2}}{a} > 0 \ \Longrightarrow$$

$$\mathrm{e}^t = \frac{x-\sqrt{x^2-a^2}}{a} \ \Rightarrow\ |t| = \ln\left|\frac{x-\sqrt{x^2-a^2}}{a}\right| = \ln\left|\frac{x^2-(x^2-a^2)}{a\left(x+\sqrt{x^2-a^2}\right)}\right| = \ln\left|\frac{a}{x+\sqrt{x^2-a^2}}\right| < \ln\left|\frac{a}{a}\right| = \ln 1 = 0 \quad ☹$$

$$\mathrm{e}^t = \frac{x+\sqrt{x^2-a^2}}{a} \ \Rightarrow\ t = \ln\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) \ \Rightarrow\ \mathrm{e}^{-2t} = \frac{a^2}{\left(x+\sqrt{x^2-a^2}\right)^2} = \frac{a^2\left(x-\sqrt{x^2-a^2}\right)^2}{[x^2-(x^2-a^2)]^2} = \frac{\left(x-\sqrt{x^2-a^2}\right)^2}{a^2}$$

$$= \frac{\left(x+\sqrt{x^2-a^2}\right)^2}{8} - \frac{a^2}{2}\ln\left|\frac{x+\sqrt{x^2-a^2}}{a}\right| - \frac{\left(x-\sqrt{x^2-a^2}\right)^2}{8} + c =$$

$$= \frac{x^2+2x\sqrt{x^2-a^2}+x^2-a^2}{8} - \frac{x^2-2x\sqrt{x^2-a^2}+x^2-a^2}{8} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + \frac{a^2}{2}\ln a + c =$$

$$= \frac{4x\sqrt{x^2-a^2}}{8} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c, \qquad \text{for } a>0,\ x>a.$$

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \int \sqrt{x^2-(-a)^2}\,\mathrm{d}x = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c, \qquad \text{for } a<0 \text{ [i.e. } -a>0,\ (-a)^2=a^2],\ x<a.$$

$$\Longrightarrow \int \sqrt{a^2+x^2}\,\mathrm{d}x = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c, \qquad \text{for } a\in R-\{0\},\ x\in(-\infty\,;-|a|)\cup(|a|\,;\infty).$$

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \int \frac{t^2+a^2}{2t}\,\frac{t^2+a^2}{2t^2}\,\mathrm{d}t = \int \frac{t^4+2t^2a^2+a^4}{4t^3}\,\mathrm{d}t = \frac{1}{4}\int \left[t+2a^2t^{-1}+a^4t^{-3}\right]\mathrm{d}t =$$

1-st Euler: $\sqrt{x^2-a^2} = t-x \ \Longrightarrow\ x^2-a^2 = t^2-2tx+x^2 \ \Longrightarrow\ x = \dfrac{t^2+a^2}{2t}, \quad t = x+\sqrt{x^2-a^2}$

$$\sqrt{x^2-a^2} = t - \frac{t^2+a^2}{2t} = \frac{t^2-a^2}{2t}, \qquad \mathrm{d}x = \frac{2t\cdot 2t - 2(t^2-a^2)}{4t^2}\,\mathrm{d}t = \frac{2t^2+2a^2}{4t^2}\,\mathrm{d}t = \frac{t^2+a^2}{2t^2}\,\mathrm{d}t$$

$$t^2 = \left(x+\sqrt{x^2-a^2}\right)^2, \quad t^{-2} = \frac{1}{\left(x+\sqrt{x^2-a^2}\right)^2}\,\frac{\left(x-\sqrt{x^2-a^2}\right)^2}{\left(x-\sqrt{x^2-a^2}\right)^2} = \frac{\left(x-\sqrt{x^2-a^2}\right)^2}{[x^2-(x^2-a^2)]^2} = \frac{\left(x-\sqrt{x^2-a^2}\right)^2}{a^4}$$

$$= \frac{1}{4}\left[\frac{t^2}{2} + 2a^2\ln|t| + a^4\frac{t^{-2}}{-2}\right] + c = \frac{\left(x+\sqrt{x^2-a^2}\right)^2}{8} - \frac{\left(x-\sqrt{x^2-a^2}\right)^2}{8} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c =$$

$$= \frac{x^2+2x\sqrt{x^2-a^2}+x^2-a^2}{8} - \frac{x^2-2x\sqrt{x^2-a^2}+x^2-a^2}{8} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c =$$

$$= \frac{4x\sqrt{x^2-a^2}}{8} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c,$$

for $a\in R-\{0\},\ x\in(-\infty\,;-|a|)\cup(|a|\,;\infty)$.

$$\int \sqrt{x^2-a^2}\,\mathrm{d}x = \int \frac{x^2-a^2}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \int \frac{x^2\,\mathrm{d}x}{\sqrt{x^2-a^2}} - \int \frac{a^2\,\mathrm{d}x}{\sqrt{x^2-a^2}} = \boxed{\begin{array}{ll} u=x & \Rightarrow\ u'=1 \\ v'=\dfrac{x}{\sqrt{x^2-a^2}} & \Rightarrow\ v=\sqrt{x^2-a^2} \end{array}} =$$

$$= \left[x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,\mathrm{d}x\right] - \int \frac{a^2\,\mathrm{d}x}{\sqrt{x^2-a^2}} = x\sqrt{x^2-a^2} - \int \sqrt{x^2-a^2}\,\mathrm{d}x - a^2 \ln\left|x+\sqrt{x^2-a^2}\right|$$

$$\xRightarrow{\text{(i.e.equation)}} x\sqrt{x^2-a^2} - a^2\ln\left|x+\sqrt{x^2-a^2}\right| + c_0 = \int \sqrt{x^2-a^2}\,\mathrm{d}x + \int \sqrt{x^2-a^2}\,\mathrm{d}x = 2\int \sqrt{x^2-a^2}\,\mathrm{d}x$$

$$\Longrightarrow \int \sqrt{x^2-a^2}\,\mathrm{d}x = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c, \qquad \text{for } a\in R-\{0\},\ x\in(-\infty\,;\,-|a|)\cup(|a|\,;\,\infty).$$

$$\int \operatorname{arctg} x\,\mathrm{d}x = \boxed{\begin{array}{ll} u'=1 & \Rightarrow\ u=x \\ v=\operatorname{arctg} x & \Rightarrow\ v'=\dfrac{1}{1+x^2} \end{array}} = x\operatorname{arctg} x - \int \frac{x}{1+x^2}\,\mathrm{d}x = x\operatorname{arctg} x - \frac{1}{2}\int \frac{2x}{1+x^2}\,\mathrm{d}x = x\operatorname{arctg} x - \frac{1}{2}\ln\left|1+x^2\right| + c =$$

$$= x\operatorname{arctg} x - \frac{1}{2}\ln\left(1+x^2\right) + c = x\operatorname{arctg} x - \ln\sqrt{1+x^2} + c, \qquad \text{for } x\in R.$$

$$I_n =: \int x^n\,\mathrm{e}^x\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x^n & \Rightarrow\ u'=nx^{n-1} \\ v'=\mathrm{e}^x & \Rightarrow\ v=\mathrm{e}^x \end{array}} = x^n\,\mathrm{e}^x - \int nx^{n-1}\,\mathrm{e}^x\,\mathrm{d}x = x^n\,\mathrm{e}^x - n\int x^{n-1}\,\mathrm{e}^x\,\mathrm{d}x = x^n\,\mathrm{e}^x - nI_{n-1}, \ \text{ for } n\in N,\ x\in R.$$

$$I_0 = \int x^0\,\mathrm{e}^x\,\mathrm{d}x = \int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x + c,$$

$$I_1 = x^1\,\mathrm{e}^x - 1I_0 = x\,\mathrm{e}^x - 1\,\mathrm{e}^x + c,$$

$$I_2 = x^2\,\mathrm{e}^x - 2I_1 = x^2\,\mathrm{e}^x - 2\left(x\,\mathrm{e}^x - 1\,\mathrm{e}^x\right) + c = x^2\,\mathrm{e}^x - 2x\,\mathrm{e}^x + 2\cdot 1\,\mathrm{e}^x + c,$$

$$I_3 = x^3\,\mathrm{e}^x - 3I_2 = x^3\,\mathrm{e}^x - 3\left(x^2\,\mathrm{e}^x - 2x\,\mathrm{e}^x + 2\cdot 1\,\mathrm{e}^x\right) + c = x^3\,\mathrm{e}^x - 3x^2\,\mathrm{e}^x + 3\cdot 2x\,\mathrm{e}^x - 3\cdot 2\cdot 1\,\mathrm{e}^x + c,$$

$$\cdots\cdots\cdots$$

$$I_n = x^n\,\mathrm{e}^x - nI_{n-1} = \sum_{j=0}^{n}(-1)^j n(n-1)\cdots(n-j+1)x^{n-j}\,\mathrm{e}^x = \mathrm{e}^x\sum_{j=0}^{n}(-1)^j n(n-1)\cdots(n-j+1)x^{n-j} = \mathrm{e}^x\sum_{j=0}^{n}\frac{(-1)^j x^{n-j} n!}{(n-j)!} =$$

$$= \mathrm{e}^x\left[x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^j n(n-1)\cdots(n-j+1)x^{n-j} + \cdots + (-1)^n n!\right] + c, \qquad \text{for } n\in N,\ x\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x+1}+\sqrt[3]{x+1}} = \int \frac{\mathrm{d}x}{\left(\sqrt[6]{x+1}\right)^3 + \left(\sqrt[6]{x+1}\right)^2} = \boxed{\begin{array}{c} \sqrt[6]{x+1}=t \\ x+1=t^6 \\ \mathrm{d}x = 6t^5\,\mathrm{d}t \end{array}} = \int \frac{6t^5\,\mathrm{d}t}{t^3+t^2} = 6\int \frac{t^3\,\mathrm{d}t}{t+1} = 6\int \frac{t^3+t^2-t^2-t+t+1-1}{t+1}\,\mathrm{d}t =$$

$$= 6\int\left[\frac{t^3+t^2}{t+1} - \frac{t^2+t}{t+1} + \frac{t+1}{t+1} - \frac{1}{t+1}\right]\mathrm{d}t = 6\int\left[t^2 - t + 1 - \frac{1}{t+1}\right]\mathrm{d}t = 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right] + c =$$

$$= 2\left(\sqrt[6]{x+1}\right)^3 - 3\left(\sqrt[6]{x+1}\right)^2 + 6\sqrt[6]{x+1} - 6\ln\left|1+\sqrt[6]{x+1}\right| + c =$$

$$= 2\sqrt{x+1} - 3\sqrt[3]{x+1} + 6\sqrt[6]{x+1} - 6\ln\left|1+\sqrt[6]{x+1}\right| + c, \qquad \text{for } x > -1.$$

$$\int \ln\left(1+x^2\right)\mathrm{d}x = \boxed{\begin{array}{ll} u'=1 & \Rightarrow\ u=x \\ v=\ln\left(1+x^2\right) & \Rightarrow\ v'=\dfrac{2x}{1+x^2}\end{array}} = x\ln\left(1+x^2\right) - \int \frac{2x^2\,\mathrm{d}x}{1+x^2} = x\ln\left(1+x^2\right) - 2\int \frac{x^2+1-1}{1+x^2}\,\mathrm{d}x =$$

$$= x\ln\left(1+x^2\right) - 2\int \left[1-\frac{1}{1+x^2}\right]\mathrm{d}x = x\ln\left(1+x^2\right) - 2x + 2\operatorname{arctg} x + c, \qquad \text{for } x\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int \frac{2t}{t^2+a^2}\,\frac{t^2+a^2}{2t^2}\,\mathrm{d}t = \int \frac{\mathrm{d}t}{t} = \ln|t|+c = \ln\left|x+\sqrt{x^2+a^2}\right|+c = \ln\left(x+\sqrt{x^2+a^2}\right)+c, \qquad \text{for } a\in R-\{0\},\ x\in R.$$

1-st Euler: $\sqrt{x^2+a^2} = t-x \implies x^2+a^2 = t^2-2tx+x^2 \implies 2tx = t^2-a^2 \implies x = \dfrac{t^2-a^2}{2t}$

$$\sqrt{x^2+a^2} = t - \frac{t^2-a^2}{2t} = \frac{t^2+a^2}{2t}, \qquad \mathrm{d}x = \frac{2t\cdot 2t - 2(t^2-a^2)}{4t^2}\,\mathrm{d}t = \frac{2t^2+2a^2}{4t^2}\,\mathrm{d}t = \frac{t^2+a^2}{2t^2}\,\mathrm{d}t$$

$$\sqrt{a^2+x^2} > \sqrt{x^2} \geq x \implies \sqrt{a^2+x^2}-x>0 \implies \ln\left|x+\sqrt{a^2+x^2}\right| = \ln\left(x+\sqrt{a^2+x^2}\right)$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int \frac{\cos t}{a}\,\frac{a\,\mathrm{d}t}{\cos^2 t} = \int \frac{\cos t\,\mathrm{d}t}{1-\sin^2 t} = \boxed{\begin{array}{c} u=\sin t \\ \mathrm{d}u=\cos t\,\mathrm{d}t\end{array}} = \int \frac{\mathrm{d}u}{1-u^2} = -\int \frac{\mathrm{d}u}{u^2-1} = -\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c_0 =$$

$$x = a\operatorname{tg} t \implies \mathrm{d}x = \frac{a\,\mathrm{d}t}{\cos^2 t} \implies \sqrt{x^2+a^2} = \sqrt{\frac{a^2\sin^2 t}{\cos^2 t}+a^2} = \sqrt{\frac{a^2\sin^2 t + a^2\cos^2 t}{\cos^2 t}} = \sqrt{\frac{a^2}{\cos^2 t}} = \frac{|a|}{\cos t} = \frac{a}{\cos t}$$

$$x\in R \implies t = \operatorname{arctg} x \in \left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right) \implies \sqrt{\cos^2 x} = |\cos x| = \cos x, \quad \operatorname{tg}^2 t = \frac{\sin^2 t}{\cos^2 t} = \frac{\sin^2 t}{1-\sin^2 t} \implies \sin^2 t = \frac{\operatorname{tg}^2 t}{\operatorname{tg}^2 t+1}$$

$$\operatorname{sgn}\sin t = \operatorname{sgn}\operatorname{tg} t \implies \sin t = \frac{\operatorname{tg} t}{\sqrt{\operatorname{tg}^2 t+1}} = \frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+1}} = \frac{x}{\sqrt{x^2+a^2}} \implies \sin t \pm 1 = \frac{x\pm\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}$$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + c_0 = \frac{1}{2}\ln\left|\frac{\sin t+1}{\sin t-1}\right| + c_0 = \frac{1}{2}\ln\left|\frac{x+\sqrt{x^2+a^2}}{x-\sqrt{x^2+a^2}}\right| + c_0 = \frac{1}{2}\ln\left|\frac{x+\sqrt{x^2+a^2}}{x-\sqrt{x^2+a^2}}\cdot\frac{x+\sqrt{x^2+a^2}}{x+\sqrt{x^2+a^2}}\right| + c_0 =$$

$$= \frac{1}{2}\ln\left|\frac{\left(x+\sqrt{x^2+a^2}\right)^2}{x^2-(x^2+a^2)}\right| + c_0 = \ln\left|x+\sqrt{x^2+a^2}\right| - \ln a + c_0 = \ln\left(x+\sqrt{x^2+a^2}\right)+c, \qquad \text{for } a>0,\ x\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \int \frac{\mathrm{d}x}{\sqrt{x^2+(-a)^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)+c, \qquad \text{for } a<0 \text{ [i.e. } -a>0,\ (-a)^2=a^2],\ x\in R.$$

$$\implies \int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\right)+c, \qquad \text{for } a\in R-\{0\},\ x\in R.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \int \frac{2t}{t^2-a^2}\,\frac{t^2-a^2}{2t^2}\,\mathrm{d}t = \int \frac{\mathrm{d}t}{t} = \ln|t|+c = \ln\left|x+\sqrt{x^2-a^2}\right|+c, \qquad \text{for } a\in R-\{0\},\ x\in(-\infty\,;-|a|)\cup(|a|\,;\infty).$$

1-st Euler: $\sqrt{x^2-a^2} = t-x \implies x^2-a^2 = t^2-2tx+x^2 \implies 2tx = t^2+a^2 \implies x = \dfrac{t^2+a^2}{2t}$

$$\sqrt{x^2-a^2} = t - \frac{t^2+a^2}{2t} = \frac{t^2-a^2}{2t}, \qquad \mathrm{d}x = \frac{2t\cdot 2t - 2(t^2+a^2)}{4t^2}\,\mathrm{d}t = \frac{2t^2-2a^2}{4t^2}\,\mathrm{d}t = \frac{t^2-a^2}{2t^2}\,\mathrm{d}t$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \boxed{\begin{matrix} x = a\sin t \\ dx = a\cos t\,dt \end{matrix}} = \int \frac{a\cos t\,dt}{a\cos t} = \int dt = t + c = \arcsin\frac{x}{a} + c, \qquad \text{for } a>0,\ x\in(-a\,;\,a).$$

$$\boxed{t = \arcsin\frac{x}{a},\quad \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2 t} = a\sqrt{1-\sin^2 t} = a\sqrt{\cos^2 t} = a\,|\cos t| = a\cos t,\quad x\in(-a\,;\,a)\ \Rightarrow\ t\in\left(-\frac{\pi}{2}\,;\,\frac{\pi}{2}\right)}$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{\sqrt{(-a)^2-x^2}} = \arcsin\frac{x}{-a} + c, \qquad \text{for } a<0 \text{ [i.e. } -a>0,\ (-a)^2=a^2],\ x\in(a\,;\,-a).$$

$$\Longrightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{|a|} + c, \qquad \text{for } a\in R-\{0\},\ x\in(-|a|\,;\,|a|).$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{t^2+1}{a(1-t^2)}\,\frac{2a(t^2-1)}{(t^2+1)^2}\,dt = -2\int\frac{dt}{t^2+1} = -2\arctg t + c = -2\arctg\frac{\sqrt{a^2-x^2}-a}{x} + c,$$

for $a\in R-\{0\},\ x\in(-|a|\,;\,|a|)-\{0\}$.

$$\boxed{\begin{matrix} \text{2-nd Euler:}\quad \sqrt{a^2-x^2} = xt + a \implies a^2-x^2 = x^2t^2+2txa+a^2 \implies -2txa = x^2t^2+x^2 \xRightarrow{x\neq 0} x = \dfrac{-2ta}{t^2+1} \\ \sqrt{a^2-x^2} = \dfrac{-2t^2a}{t^2+1}+a = \dfrac{a(1-t^2)}{t^2+1},\quad dx = \dfrac{-2a(t^2+1)+2ta\cdot 2t}{(t^2+1)^2}\,dt = \dfrac{2a(t^2-1)}{(t^2+1)^2}\,dt,\quad t = \dfrac{\sqrt{a^2-x^2}-a}{x} \end{matrix}}$$

$$\int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\,dx = \int \frac{\sqrt{1-\sqrt{x}}}{\sqrt{1+\sqrt{x}}}\cdot\frac{\sqrt{1-\sqrt{x}}}{\sqrt{1-\sqrt{x}}}\,dx = \int\frac{1-\sqrt{x}}{\sqrt{1-x}}\,dx = \int\frac{dx}{\sqrt{1-x}} - \int\frac{\sqrt{x}}{\sqrt{1-x}}\,dx = \int(1-x)^{-\frac12}\,dx - \int\sqrt{\frac{x}{1-x}}\,dx =$$

$$\boxed{\begin{matrix} 1-x=u \\ dx = -\,du \end{matrix}} \quad \text{resp.} \quad \boxed{\frac{x}{1-x} = t^2 \implies x = t^2 - xt^2 \implies x = \frac{t^2}{1+t^2} \implies dx = \frac{2t(1+t^2)-t^2\cdot 2t}{(1+t^2)^2}\,dt = \frac{2t\,dt}{(1+t^2)^2}}$$

$$= -\int u^{-\frac12}\,du - \int\frac{t\cdot 2t\,dt}{(1+t^2)^2} = -\frac{u^{\frac12}}{\frac12} - 2\int\frac{t^2\,dt}{(1+t^2)^2} = -2\sqrt{u} - 2\int\frac{1+t^2-1}{(1+t^2)^2}\,dt =$$

$$= -2\sqrt{1-x} - 2\int\left[\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\right]dt = -2\sqrt{1-x} - 2\left[\arctg t - \frac12\arctg t - \frac12\frac{t}{t^2+1}\right] + c =$$

$$\boxed{t^2+1 = \frac{x}{1-x}+1 = \frac{x+1-x}{1-x} = \frac{1}{1-x} \implies \frac{t}{t^2+1} = (1-x)\sqrt{\frac{x}{1-x}} = \sqrt{(1-x)x} = \sqrt{x-x^2}}$$

$$= -2\sqrt{1-x} - \arctg t + \frac{t}{t^2+1} + c = -2\sqrt{1-x} - \arctg\sqrt{\frac{x}{1-x}} + \sqrt{x-x^2} + c, \qquad \text{for } x\in\langle 0\,;\,1).$$

$$\int |x|\,dx = \begin{cases} \displaystyle\int x\,dx = \frac{x^2}{2} + c = \frac{x\,|x|}{2} + c, & \text{for } x\geq 0, \\ \displaystyle -\int x\,dx = -\frac{x^2}{2} + c = \frac{x\,|x|}{2} + c, & \text{for } x<0. \end{cases} \qquad \Longrightarrow \int |x|\,dx = \frac{x\,|x|}{2} + c, \qquad \text{for } x\in R.$$

$$\int x^2\,e^{2x}\,dx = \boxed{\begin{matrix} u=x^2 & \Rightarrow & u'=2x \\ v'=e^{2x} & \Rightarrow & v=\dfrac{e^{2x}}{2}\end{matrix}} = \frac{x^2\,e^{2x}}{2} - \int \frac{2x\,e^{2x}}{2}\,dx = \frac{x^2\,e^{2x}}{2} - \int x\,e^{2x}\,dx = \boxed{\begin{matrix} u=x & \Rightarrow & u'=1 \\ v'=e^{2x} & \Rightarrow & v=\dfrac{e^{2x}}{2}\end{matrix}} =$$

$$= \frac{x^2\,e^{2x}}{2} - \left[\frac{x\,e^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx\right] = \frac{x^2\,e^{2x}}{2} - \frac{x\,e^{2x}}{2} + \int \frac{e^{2x}}{2}\,dx = \frac{x^2\,e^{2x}}{2} - \frac{x\,e^{2x}}{2} + \frac{e^{2x}}{4} + c = e^{2x}\left[\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + c, \quad \text{for } x\in R.$$

$$\int x^2\,e^{2x}\,dx = e^{2x}\left[Ax^2+Bx+C\right] + c = e^{2x}\left[\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right] + c, \qquad \text{for } \ x\in R.$$

$$\boxed{\begin{array}{l} \left[\int x^2\,e^{2x}\,dx\right]' = \left[e^{2x}\,(Ax^2+Bx+C)+c\right]' \implies x^2\,e^{2x} = 2\,e^{2x}\,(Ax^2+Bx+C) + e^{2x}\,(2Ax+B) \implies \\ x^2 = 2Ax^2+2Bx+2Ax+2C+B \implies 1=2A,\ 0=2A+2B,\ 0=B+2C \implies A=\frac{1}{2},\ B=-\frac{1}{2},\ C=\frac{1}{4} \end{array}}$$

$$\int x^9\,e^{2x}\,dx = \boxed{\begin{matrix} u=x^9 & \Rightarrow & u'=9x^8 \\ v'=e^{2x} & \Rightarrow & v=\dfrac{e^{2x}}{2}\end{matrix}} = \frac{x^9\,e^{2x}}{2} - \int \frac{9x^8\,e^{2x}}{2}\,dx = \boxed{\begin{matrix} u=x^8 & \Rightarrow & u'=8x^7 \\ v'=e^{2x} & \Rightarrow & v=\dfrac{e^{2x}}{2}\end{matrix}} = \cdots.$$

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$$\int x^9\,e^{2x}\,dx = e^{2x}\left[Ax^9+Bx^8+Cx^7+Dx^6+Ex^5+Fx^4+Gx^3+Hx^2+Ix+J\right] + c$$

$$\boxed{\begin{array}{l} x^9\,e^{2x} = 2\,e^{2x}\,(Ax^9+Bx^8+Cx^7+Dx^6+Ex^5+Fx^4+Gx^3+Hx^2+Ix+J) + \\ \qquad\qquad + e^{2x}\,(9Ax^8+8Bx^7+7Cx^6+6Dx^5+5Ex^4+4Fx^3+3Gx^2+2Hx+I) \implies \\ x^9 = 2Ax^9+(2B+9A)x^8+(2C+8B)x^7+(2D+7C)x^6+(2E+6D)x^5+(2F+5E)x^4+ \\ \qquad\qquad +(2G+4F)x^3+(2H+3G)x^2+(2I+2H)x+2J+I \implies \\ A=\frac{1}{2},\ B=-\frac{9}{4},\ C=9,\ D=-\frac{63}{2},\ E=\frac{189}{2},\ F=-\frac{945}{4},\ G=\frac{945}{2},\ H=-\frac{2835}{4},\ I=\frac{2835}{4},\ J=-\frac{2835}{8} \end{array}}$$

$$\implies \int x^9\,e^{2x}\,dx = e^{2x}\left[\frac{x^9}{2} - \frac{9x^8}{4} + 9x^7 - \frac{63x^6}{2} + \frac{189x^5}{2} - \frac{945x^4}{4} + \frac{945x^3}{2} - \frac{2835x^2}{4} + \frac{2835x}{4} - \frac{2835}{8}\right] + c, \quad \text{for } x\in R.$$

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \boxed{\begin{array}{l} t=\sqrt{\dfrac{1+x}{1-x}},\ t^2=\dfrac{1+x}{1-x} \implies 1+x=t^2-xt^2 \implies x=\dfrac{t^2-1}{1+t^2},\quad x\in\langle -1\,;\,1) \\ t^2+1=\dfrac{2}{1-x},\quad dx=\dfrac{2t(1+t^2)-2t(t^2-1)}{(1+t^2)^2}\,dt=\dfrac{4t\,dt}{(1+t^2)^2},\quad t\in\langle 0\,;\,\infty) \end{array}} = \int \frac{4t^2\,dt}{(1+t^2)^2} = 4\int \frac{t^2+1-1\,dt}{(1+t^2)^2} =$$

$$= 4\int \left[\frac{1}{1+t^2}1\frac{1}{(1+t^2)^2}\right] dt = 4\left[\operatorname{arctg} t - \frac{1}{2}\operatorname{arctg} t - \frac{1}{2}\frac{t}{t^2+1}\right] + c = 2\operatorname{arctg} t - \frac{2t}{t^2+1} + c =$$

$$= 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - 2\sqrt{\frac{1+x}{1-x}}\frac{1-x}{2} + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c, \qquad \text{for } \ x\in\langle -1\,;\,1).$$

$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \boxed{\begin{array}{c} x=-t \\ dx=-\,dt \end{array}} = -\int \sqrt{\left(\frac{1-t}{1+t}\right)^3}\,dt = -\left[\sqrt{1-t^2} - 2\operatorname{arctg}\sqrt{\frac{1-t}{1+t}}\right] + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c,$$

$$\text{for } \ x\in\langle -1\,;\,1).$$

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \int \left[\frac{2t}{t^2+1}\right]^2 \frac{2t}{t^2-1}\,\frac{t^2-1}{2t^2}\,\mathrm{d}t = \int \frac{4t\,\mathrm{d}t}{(t^2+1)^2} = \boxed{\begin{array}{c} t^2+1=u \\ 2t\,\mathrm{d}t=\mathrm{d}u \end{array}} = \int \frac{2\,\mathrm{d}u}{u^2} = 2\int u^{-2}\,\mathrm{d}u =$$

$$\boxed{\begin{array}{l} \text{1-st Euler:} \quad \sqrt{x^2-1}=t-x \implies x^2-1=t^2-2tx+x^2 \implies 2tx=t^2+1 \implies x=\frac{t^2+1}{2t} \\ \sqrt{x^2-1}=t-\frac{t^2+1}{2t}=\frac{t^2-1}{2t}, \qquad \mathrm{d}x=\frac{2t\cdot 2t-2(t^2+1)}{4t^2}\,\mathrm{d}t=\frac{2t^2-2}{4t^2}\,\mathrm{d}t=\frac{t^2-1}{2t^2}\,\mathrm{d}t \end{array}}$$

$$= 2\frac{u^{-1}}{-1}+c_1 = c_1-\frac{2}{t^2+1} = c_1-\frac{2}{2tx} = c_1-\frac{1}{x\left(x+\sqrt{x^2-1}\right)}, \qquad \text{for } \ x\in(-\infty\,;\,-1)\cup(1\,;\,\infty).$$

$$= c_1-\frac{1}{x\left(x+\sqrt{x^2-1}\right)}\cdot\frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = c_1-\frac{x-\sqrt{x^2-1}}{x\,(x^2-x^2+1)} = c_1-\frac{x-\sqrt{x^2-1}}{x} = c_1-1+\frac{\sqrt{x^2-1}}{x} = c_2+\frac{\sqrt{x^2-1}}{x},$$

for $\ x\in(-\infty\,;\,-1)\cup(1\,;\,\infty)$.

$$\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-1}} = \boxed{\begin{array}{l} x=\frac{1}{t},\ t=\frac{1}{x} \implies \mathrm{d}x=(t^{-1})'\,\mathrm{d}t=-\frac{\mathrm{d}t}{t^2} \\ \sqrt{x^2-1}=\sqrt{\frac{1}{t^2}-1}=\frac{\sqrt{1-t^2}}{t} \quad \text{for } t\in(0\,;\,1) \end{array}} = -\int t^2\frac{t}{\sqrt{1-t^2}}\frac{\mathrm{d}t}{t^2} = \int\frac{-t\,\mathrm{d}t}{\sqrt{1-t^2}} = \boxed{\begin{array}{c} 1-t^2=u \\ -2t\,\mathrm{d}t=\mathrm{d}u \end{array}} =$$

$$= \frac{1}{2}\int\frac{\mathrm{d}u}{u^{\frac{1}{2}}} = \frac{1}{2}\int u^{-\frac{1}{2}}\,\mathrm{d}u = \frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c_2 = \sqrt{1-t^2}+c = \sqrt{1-\frac{1}{x^2}}+c_2 = \frac{\sqrt{x^2-1}}{x}+c_2, \qquad \text{for } \ x\in(1\,;\,\infty).$$

$$\int x^x(\ln x+1)\,\mathrm{d}x = \boxed{\begin{array}{l} u=x^x \ \Rightarrow\ u'=\left[\mathrm{e}^{\ln x^x}\right]'=\left[\mathrm{e}^{x\ln x}\right]'=\mathrm{e}^{x\ln x}\left[\ln x+\frac{x}{x}\right]=x^x(\ln x+1) \\ v'=1 \ \Rightarrow\ v=x \end{array}} = x^x+c, \qquad \text{for } \ x>0.$$

$$\int \ln\left(x+\sqrt{x^2+1}\right)\mathrm{d}x = \boxed{\begin{array}{ll} u=\ln\left(x+\sqrt{x^2+1}\right) & \Rightarrow\ u'=\frac{1}{\sqrt{x^2+1}} \\ v'=1 & \Rightarrow\ v=x \end{array}} = x\ln\left(x+\sqrt{x^2+1}\right)-\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+1}} = \boxed{\begin{array}{c} x^2+1=t \\ 2x\,\mathrm{d}x=\mathrm{d}t \end{array}} =$$

$$= x\ln\left(x+\sqrt{x^2+1}\right)-\frac{1}{2}\int\frac{\mathrm{d}t}{\sqrt{t}} = x\ln\left(x+\sqrt{x^2+1}\right)-\frac{1}{2}\int t^{-\frac{1}{2}}\,\mathrm{d}t =$$

$$= x\ln\left(x+\sqrt{x^2+1}\right)-\frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c = x\ln\left(x+\sqrt{x^2+1}\right)-\sqrt{x^2+1}+c, \qquad \text{for } \ x\in R.$$

$$\int\frac{\mathrm{d}x}{\sqrt{x-3}+\sqrt{x-4}} = \int\frac{1}{\sqrt{x-3}+\sqrt{x-4}}\,\frac{\sqrt{x-3}-\sqrt{x-4}}{\sqrt{x-3}-\sqrt{x-4}}\,\mathrm{d}x = \int\frac{\sqrt{x-3}-\sqrt{x-4}}{x-3-(x-4)}\,\mathrm{d}x = \int\left[(x-3)^{\frac{1}{2}}-(x-4)^{\frac{1}{2}}\right]\mathrm{d}x =$$

$$= \frac{(x-3)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(x-4)^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{2\sqrt{(x-3)^3}}{3}-\frac{2\sqrt{(x-4)^3}}{3}+c, \qquad \text{for } \ x>3.$$

$$\int \min_{x\in(0;\infty)}\left\{1,\frac{1}{x}\right\}\mathrm{d}x = \begin{cases} \displaystyle\int\mathrm{d}x = x+c_1, & \text{for } \ x\in(0\,;\,1\rangle\,, \\ \displaystyle\int\frac{\mathrm{d}x}{x} = \ln x+c_2, & \text{for } \ x\in\langle 1\,;\,\infty)\,. \end{cases}$$

$$\int \frac{2x^2-x+1}{x\sqrt{1+x-x^2}}\,\mathrm{d}x = \int \frac{2x-1}{\sqrt{1+x-x^2}}\,\mathrm{d}x + \int \frac{\mathrm{d}x}{x\sqrt{1+x-x^2}} = \int \frac{-\mathrm{d}s}{\sqrt{s}} - \int t\frac{t}{\sqrt{t^2+t-1}}\frac{\mathrm{d}t}{t^2} = -\int s^{-\frac{1}{2}}\,\mathrm{d}s - \int \frac{\mathrm{d}t}{\sqrt{\left(t+\frac{1}{2}\right)^2-\frac{5}{4}}} =$$

$$\boxed{\begin{array}{c} 1+x-x^2=s \\ (1-2x)\,\mathrm{d}x=\mathrm{d}s \end{array}}$$

$$\boxed{\begin{array}{l} x=\frac{1}{t},\ t=\frac{1}{x} \implies \mathrm{d}x=-\frac{\mathrm{d}t}{t^2} \implies \sqrt{1+x-x^2}=\sqrt{1+\frac{1}{t}-\frac{1}{t^2}}=\frac{\sqrt{t^2+t-1}}{t} \quad \text{for } t\notin\left\langle\frac{-1-\sqrt{5}}{2};\frac{-1+\sqrt{5}}{2}\right\rangle \\ x\in\left(\frac{1-\sqrt{5}}{2};\frac{1+\sqrt{5}}{2}\right)-\{0\} \implies \begin{cases} 0<x<\frac{1+\sqrt{5}}{2} \implies t=\frac{1}{x}>\frac{2}{1+\sqrt{5}}=\frac{2(1-\sqrt{5})}{1-5}=\frac{\sqrt{5}-1}{2}>0 \\ \frac{1-\sqrt{5}}{2}<x<0 \implies t=\frac{1}{x}<\frac{2}{1-\sqrt{5}}=\frac{2(1+\sqrt{5})}{1-5}=-\frac{\sqrt{5}+1}{2}<0 \end{cases} \\ t^2+t-1=\left(t+\frac{1}{2}\right)^2-1-\frac{1}{4}=\left(t+\frac{1}{2}\right)^2-\frac{5}{4}>0 \implies \left|t+\frac{1}{2}\right|>\frac{\sqrt{5}}{2} \implies t+\frac{1}{2}<-\frac{\sqrt{5}}{2} \text{ or } \frac{\sqrt{5}}{2}<t+\frac{1}{2} \end{array}}$$

$$= \boxed{\begin{array}{c} t+1/2=u \\ \mathrm{d}t=\mathrm{d}u \end{array}} = -\frac{s^{\frac{1}{2}}}{\frac{1}{2}} - \int \frac{\mathrm{d}u}{\sqrt{u^2-\frac{5}{4}}} = -2\sqrt{s} - \ln\left|u+\sqrt{u^2-\frac{5}{4}}\right| + c = -2\sqrt{1+x-x^2} - \ln\left|t+\frac{1}{2}+\sqrt{t^2+t-1}\right| + c =$$

$$= -2\sqrt{1+x-x^2} - \ln\left|\frac{1}{x}+\frac{1}{2}+\sqrt{\frac{1}{x^2}+\frac{1}{x}-1}\right| + c = -2\sqrt{1+x-x^2} - \ln\left|\frac{2+x}{2x}+\frac{\sqrt{1+x-x^2}}{x}\right| + c =$$

$$= -2\sqrt{1+x-x^2} - \ln\left|\frac{2+x+2\sqrt{1+x-x^2}}{2x}\right| + c, \qquad \text{for } x\in\left(\frac{1-\sqrt{5}}{2};\frac{1+\sqrt{5}}{2}\right)-\{0\}.$$

$$\int \frac{2x^2-x+1}{x\sqrt{1+x-x^2}}\,\mathrm{d}x = \int \frac{t^4+2t^3+9t^2-6t+2}{(t^2+1)^2}\cdot\frac{t^2+1}{1-2t}\cdot\frac{t^2+1}{-t^2+t+1}\cdot\frac{2t^2-2t-2}{(t^2+1)^2}\,\mathrm{d}t = 2\int \frac{t^4+2t^3+9t^2-6t+2}{(2t-1)(t^2+1)^2}\,\mathrm{d}t =$$

$$\boxed{\begin{array}{l} \text{2-nd Euler:} \quad \sqrt{-x^2+x+1}=xt+1 \implies -x^2+x+1=x^2t^2+2tx+1 \implies x-2tx=x^2t^2+x^2 \implies 1-2t=xt^2+x \\ \implies x=\frac{1-2t}{t^2+1}, \quad \sqrt{1+x-x^2}=xt+1=\frac{1-2t}{t^2+1}t+1=\frac{t-2t^2+t^2+1}{t^2+1}=\frac{-t^2+t+1}{t^2+1}, \quad t=\frac{\sqrt{1+x-x^2}-1}{x} \\ \mathrm{d}x=\frac{-2(t^2+1)-2t(1-2t)}{(t^2+1)^2}\,\mathrm{d}t=\frac{-2t^2-2-2t+4t^2}{(t^2+1)^2}\,\mathrm{d}t=\frac{2t^2-2t-2}{(t^2+1)^2}\,\mathrm{d}t, \quad t^2=\frac{-x^2+x+2-2\sqrt{1+x-x^2}}{x^2} \\ 2x^2-x+1=2\left[\frac{1-2t}{t^2+1}\right]^2-\frac{1-2t}{t^2+1}+1=\frac{2(1-4t+4t^2)}{(t^2+1)^2}-\frac{t^2+1-2t^3-2t}{(t^2+1)^2}+\frac{t^4+2t^2+1}{(t^2+1)^2}=\frac{t^4+2t^3+9t^2-6t+2}{(t^2+1)^2} \\ 4+2t=\frac{4x+2\sqrt{1+x-x^2}-2}{x}, \quad t^2+1=\frac{-x^2+x+2-2\sqrt{1+x-x^2}+x^2}{x^2}=\frac{x+2-2\sqrt{1+x-x^2}}{x^2} \end{array}}$$

$$= 2\int\left[\frac{1}{2t-1}+\frac{1}{t^2+1}+\frac{4t-2}{(t^2+1)^2}\right]\mathrm{d}t = \int\frac{2\,\mathrm{d}t}{2t-1}+2\int\frac{\mathrm{d}t}{t^2+1}+4\int\frac{2t}{(t^2+1)^2}\,\mathrm{d}t-4\int\frac{1}{(t^2+1)^2}\,\mathrm{d}t =$$

$$\boxed{\frac{t^4+2t^3+9t^2-6t+2}{(2t-1)(t^2+1)^2}=\frac{A}{2t-1}+\frac{Bt+C}{t^2+1}+\frac{Dt+E}{(t^2+1)^2} \implies A=1,\ B=0,\ C=1,\ D=4,\ E=-2} \quad \boxed{\begin{array}{c} 2t-1=u \\ 2\,\mathrm{d}t=\mathrm{d}u \end{array}} \quad \boxed{\begin{array}{c} t^2+1=v \\ 2t\,\mathrm{d}t=\mathrm{d}v \end{array}}$$

$$= \int\frac{\mathrm{d}u}{u}+2\int\frac{\mathrm{d}t}{t^2+1}+4\int\frac{\mathrm{d}v}{v^2}-4\int\frac{\mathrm{d}t}{(t^2+1)^2} \overset{\text{page}}{\underset{17}{=}} \ln|u|+2\operatorname{arctg}t+4\frac{v^{-1}}{-1}-4\left[\frac{1}{2}\operatorname{arctg}t+\frac{1}{2}\frac{t}{t^2+1}\right]+c =$$

$$\boxed{\begin{array}{l} 1+x-x^2>0 \implies x^2-x-1=\left(x-\frac{1}{2}\right)^2-\frac{1}{4}-1=\left(x-\frac{1}{2}\right)^2-\frac{5}{4}<0 \implies \left(x-\frac{1}{2}\right)^2<\frac{5}{4} \implies \\ \left|x-\frac{1}{2}\right|<\frac{\sqrt{5}}{2} \implies -\frac{\sqrt{5}}{2}<x-\frac{1}{2}<\frac{\sqrt{5}}{2} \implies \frac{1}{2}-\frac{\sqrt{5}}{2}<x<\frac{1}{2}+\frac{\sqrt{5}}{2} \implies x\in\left(\frac{1-\sqrt{5}}{2};\frac{1+\sqrt{5}}{2}\right) \end{array}}$$

$$= \ln|2t-1|+2\operatorname{arctg}t-\frac{4}{v}-2\operatorname{arctg}t-\frac{2t}{t^2+1}+c = \ln|2t-1|-\frac{4}{t^2+1}-\frac{2t}{t^2+1}+c = \ln|2t-1|-\frac{4+2t}{t^2+1}+c =$$

$$= \ln\left|\frac{2\sqrt{1+x-x^2}-2-x}{x}\right| - x\frac{4x-2+2\sqrt{1+x-x^2}}{x+2-2\sqrt{1+x-x^2}}+c, \qquad \text{for } x\in\left(\frac{1-\sqrt{5}}{2};\frac{1+\sqrt{5}}{2}\right)-\{0\}.$$

$$\int \frac{\sqrt{x^2-x}}{x}\,\mathrm{d}x = \int \frac{\sqrt{x^2-x}}{\sqrt{x^2}}\,\mathrm{d}x = \int \sqrt{\frac{x^2-x}{x^2}}\,\mathrm{d}x = \int \sqrt{\frac{x-1}{x}}\,\mathrm{d}x = \int \frac{t\cdot 2t\,\mathrm{d}t}{(1-t^2)^2} = \int \frac{2t^2\,\mathrm{d}t}{(1-t^2)^2} =$$

$$\boxed{x\neq 0,\ \ x^2-x=x(x-1)\geq 0 \implies x\neq 0,\ \ x\in(-\infty;0\rangle\cup\langle 1;\infty) \implies x\in(-\infty;0)\cup\langle 1;\infty)}$$

$$x\geq 0:\quad \sqrt{\frac{x-1}{x}}=t,\ \ t\in\langle 0;1) \implies \frac{x-1}{x}=t^2 \implies x=\frac{1}{1-t^2} \implies \mathrm{d}x=\frac{-(-2t)\,\mathrm{d}t}{(1-t^2)^2}=\frac{2t\,\mathrm{d}t}{(1-t^2)^2}$$

$$1\pm t = 1\pm\frac{\sqrt{x-1}}{\sqrt{x}}=\frac{\sqrt{x}\pm\sqrt{x-1}}{\sqrt{x}},\qquad \frac{t}{1-t^2}=xt=x\sqrt{\frac{x-1}{x}}=\sqrt{x}\sqrt{x-1}=\sqrt{x^2-x}$$

$$\frac{1-t}{1+t}=\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}=\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}+\sqrt{x-1}}\,\frac{\sqrt{x}-\sqrt{x-1}}{\sqrt{x}-\sqrt{x-1}}=\frac{x-2\sqrt{x(x-1)}+x-1}{x-(x-1)}=2x-1-2\sqrt{x^2-x}>0$$

$$=\frac{1}{2}\int\left[\frac{1}{t-1}+\frac{1}{(t-1)^2}-\frac{1}{t+1}+\frac{1}{(t+1)^2}\right]\mathrm{d}t=\frac{1}{2}\left[\ln|t-1|+\frac{(t-1)^{-1}}{-1}-\ln|t+1|+\frac{(t+1)^{-1}}{-1}\right]+c=$$

$$\boxed{\frac{2t^2}{(1-t^2)^2}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2} \implies A=\frac{1}{2},\ B=\frac{1}{2},\ C=-\frac{1}{2},\ D=\frac{1}{2}}$$

$$=\frac{1}{2}\left[\ln\frac{1-t}{1+t}+\frac{1}{1-t}-\frac{1}{1+t}\right]+c=\frac{1}{2}\left[\ln\frac{1-t}{1+t}+\frac{1+t-(1-t)}{1-t^2}\right]+c=\frac{1}{2}\ln\frac{1-t}{1+t}+\frac{t}{1-t^2}+c=$$

$$=\frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x}+c,\qquad \text{for}\ \ x\in\langle 1;\infty).$$

$$\int \frac{\sqrt{x^2-x}}{x}\,\mathrm{d}x = \int \frac{\sqrt{x^2-x}}{-\sqrt{x^2}}\,\mathrm{d}x = -\int \sqrt{\frac{x-1}{x}}\,\mathrm{d}x = -\int \frac{2t^2\,\mathrm{d}t}{(1-t^2)^2} = -\frac{1}{2}\ln\frac{1-t}{1+t}-\frac{t}{1-t^2}+c=$$

$$x<0:\quad \sqrt{\frac{x-1}{x}}=t,\ \ t\in\langle 0;1) \implies \frac{x-1}{x}=t^2 \implies x=\frac{1}{1-t^2} \implies \mathrm{d}x=\frac{2t\,\mathrm{d}t}{(1-t^2)^2}$$

$$1\pm t=1\pm\frac{\sqrt{1-x}}{\sqrt{-x}}=\frac{\sqrt{-x}\pm\sqrt{1-x}}{\sqrt{-x}},\qquad \frac{t}{1-t^2}=xt=-\sqrt{(-x)^2}\sqrt{\frac{1-x}{-x}}=-\sqrt{-x}\sqrt{1-x}=-\sqrt{x^2-x}$$

$$\frac{1+t}{1-t}=\frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}-\sqrt{1-x}}=\frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}-\sqrt{1-x}}\,\frac{\sqrt{-x}+\sqrt{1-x}}{\sqrt{-x}+\sqrt{1-x}}=\frac{-x+2\sqrt{-x(1-x)}+1-x}{-x-(1-x)}=2x-1-2\sqrt{x^2-x}>0$$

$$=\frac{1}{2}\ln\frac{1+t}{1-t}-\frac{t}{1-t^2}+c=\frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x}+c,\qquad \text{for}\ \ x\in(-\infty;0).$$

$$\implies \int \frac{\sqrt{x^2-x}}{x}\,\mathrm{d}x=\frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x}+c,\qquad \text{for}\ \ x\in(-\infty;0)\cup\langle 1;\infty).$$

$$\int \frac{x^5\,\mathrm{d}x}{\sqrt{x^3+1}}=\frac{1}{3}\int\frac{x^3\cdot 3x^2\,\mathrm{d}x}{\sqrt{x^3+1}}=\boxed{\begin{matrix}x^3+1=t\\ 3x^2\,\mathrm{d}x=\mathrm{d}t\end{matrix}}=\frac{1}{3}\int\frac{(t-1)\,\mathrm{d}t}{\sqrt{t}}=\frac{1}{3}\int\left[\sqrt{t}-\frac{1}{\sqrt{t}}\right]\mathrm{d}t=\frac{1}{3}\int\left[t^{\frac{1}{2}}-t^{-\frac{1}{2}}\right]\mathrm{d}t=\frac{1}{3}\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}-\frac{t^{\frac{1}{2}}}{\frac{1}{2}}\right]+c=$$

$$=\frac{2\sqrt{t^3}}{9}-\frac{2\sqrt{t}}{3}+c=\frac{2\sqrt{(x^3+1)^3}}{9}-\frac{2\sqrt{x^3+1}}{3}+c,\qquad \text{for}\ \ x\in(-1;\infty).$$

$$\int \frac{dx}{\sqrt{5+4e^x}} = \boxed{\begin{matrix} e^x = t,\ t>0 \\ x=\ln t,\ dx=\frac{dt}{t}\end{matrix}} = \int \frac{dt}{t\sqrt{5+4t}} = \boxed{\begin{matrix} 4t=u^2 \\ 4\,dt = 2u\,du\end{matrix}} = \int \frac{4}{u^2\sqrt{5+u^2}}\frac{u\,du}{2} = 2\int \frac{du}{u\sqrt{5+u^2}} =$$

$$= \boxed{\begin{matrix} u=\frac{\sqrt5}{v} \implies du = -\frac{\sqrt5\,dv}{v^2}, \quad u>0 \\ \sqrt{5+u^2}=\sqrt{5+\frac{5}{v^2}}=\frac{\sqrt5}{v}\sqrt{v^2+1}, \quad v>0\end{matrix}} = 2\int \frac{v}{\sqrt5}\frac{v}{\sqrt5\sqrt{v^2+1}}\frac{-\sqrt5\,dv}{v^2} = -\frac{2}{\sqrt5}\int\frac{dv}{\sqrt{v^2+1}} =$$

$$= -\frac{2}{\sqrt5}\ln\left(v+\sqrt{v^2+1}\right)+c_1 = -\frac{2}{\sqrt5}\ln\left(\frac{\sqrt5}{u}+\sqrt{\frac{5}{u^2}+1}\right)+c_1 = -\frac{2}{\sqrt5}\ln\left(\frac{\sqrt5}{u}+\frac{\sqrt{5+u^2}}{u}\right)+c_1 =$$

$$= -\frac{2}{\sqrt5}\ln\frac{\sqrt5+\sqrt{5+u^2}}{u}+c_1 = \frac{2}{\sqrt5}\left[\ln u - \ln\left(\sqrt5+\sqrt{5+u^2}\right)\right]+c_1 = \frac{2}{\sqrt5}\left[\ln(4t)^{\frac12} - \ln\left(\sqrt5+\sqrt{5+4t}\right)\right]+c_1 =$$

$$= \frac{2}{\sqrt5}\left[\frac12\ln(4e^x) - \ln\left(\sqrt5+\sqrt{5+4e^x}\right)\right]+c_1 = \frac{2}{\sqrt5}\left[\frac12\ln4+\frac12\ln e^x - \ln\left(\sqrt5+\sqrt{5+4e^x}\right)\right]+c_1 =$$

$$= \frac{\ln4}{\sqrt5}+\frac{x}{\sqrt5}-\frac{2}{\sqrt5}\ln\left(\sqrt5+\sqrt{5+4e^x}\right)+c_1 = \frac{x}{\sqrt5}-\frac{2}{\sqrt5}\ln\left(\sqrt5+\sqrt{5+4e^x}\right)+c, \qquad \text{for } x\in R.$$

$$\int \frac{dx}{\sqrt{5+4e^x}} = \boxed{\begin{matrix} e^x=t,\ 4t=u^2,\ u=\frac{\sqrt5}{v} \implies e^x=t=\frac{u^2}{4}=\frac{5}{4v^2},\ v^2=\frac{5}{4e^x} \\ v=\frac{\sqrt5}{2\sqrt{e^x}},\quad x=\ln\frac{5}{4v^2}=\ln5-\ln4-2\ln v \implies dx=\frac{-2\,dv}{v}\end{matrix}} = \int \frac{1}{\sqrt{5+4\frac{5}{4v^2}}}\frac{-2\,dv}{v} =$$

$$= -\frac{2}{\sqrt5}\int\frac{dv}{\sqrt{v^2+1}} = -\frac{2}{\sqrt5}\ln\left(v+\sqrt{v^2+1}\right)+c_1 = -\frac{2}{\sqrt5}\ln\left(\frac{\sqrt5}{2\sqrt{e^x}}+\sqrt{\frac{5}{4e^x}+1}\right)+c_1 =$$

$$= -\frac{2}{\sqrt5}\ln\frac{\sqrt5+\sqrt{5+4e^x}}{2\sqrt{e^x}}+c_1 = -\frac{2}{\sqrt5}\ln\frac{\sqrt5+\sqrt{5+4e^x}}{2\,(e^x)^{\frac12}}+c_1 = -\frac{2}{\sqrt5}\left[\ln\left(\sqrt5+\sqrt{5+4e^x}\right)-\ln2-\frac12\ln e^x\right]+c_1 =$$

$$= \frac{2\ln2}{\sqrt5}+\frac{x}{\sqrt5}-\frac{2}{\sqrt5}\ln\left(\sqrt5+\sqrt{5+4e^x}\right)+c_1 = \frac{x}{\sqrt5}-\frac{2}{\sqrt5}\ln\left(\sqrt5+\sqrt{5+4e^x}\right)+c, \qquad \text{for } x\in R.$$

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \boxed{\begin{matrix} u=\sqrt{1-x^2} \ \Rightarrow\ u'=\frac12(1-x^2)^{-\frac12}(-2x)=\frac{-x}{\sqrt{1-x^2}} \\ v'=\frac{1}{x^2}=x^{-2} \ \Rightarrow\ v=\frac{x^{-1}}{-1}=-\frac1x\end{matrix}} = -\frac{\sqrt{1-x^2}}{x}-\int\frac{-x}{\sqrt{1-x^2}}\frac{-\,dx}{x} =$$

$$= -\frac{\sqrt{1-x^2}}{x}-\int\frac{dx}{\sqrt{1-x^2}} = -\frac{\sqrt{1-x^2}}{x}-\arcsin x+c, \qquad \text{for } x\in\langle -1\,;1\rangle-\{0\}.$$

$$\int \frac{\sqrt{1-x^2}}{x^2}\,dx = \boxed{\begin{matrix} x=\sin t,\quad t=\arcsin x \implies dx=\cos t\,dt, \qquad x\in\langle -1\,;1\rangle \\ \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t, \quad t\in\left\langle -\frac{\pi}{2}\,;\frac{\pi}{2}\right\rangle\end{matrix}} = \int\frac{\cos t\cdot\cos t\,dt}{\sin^2 t} = \int\frac{1-\sin^2 t\,dt}{\sin^2 t} =$$

$$= \int\left[\frac{1}{\sin^2 t}-1\right]dt = -\operatorname{cotg} t - t + c = c-\frac{\cos t}{\sin t}-t = c-\frac{\sqrt{1-x^2}}{x}-\arcsin x, \qquad \text{for } x\in\langle -1\,;1\rangle-\{0\}.$$

$$\int\sqrt{\left(\frac{1-x}{1+x}\right)^3}\,\mathrm{d}x=\boxed{\begin{array}{l} t=\sqrt{\frac{1-x}{1+x}},\ t^2=\frac{1-x}{1+x}\implies 1-x=t^2+xt^2\implies x=\frac{1-t^2}{1+t^2},\ x\in(-1;1\rangle \\ t^2+1=\frac{2}{1+x},\quad \mathrm{d}x=\frac{-2t(1+t^2)-2t(1-t^2)}{(1+t^2)^2}\,\mathrm{d}t=\frac{-4t\,\mathrm{d}t}{(1+t^2)^2},\quad t\in\langle 0;\infty)\end{array}}=\int\frac{-4t^4\,\mathrm{d}t}{(1+t^2)^2}=$$

$$\boxed{\frac{t^4}{(1+t^2)^2}=\frac{t^4+2t^2+1-2t^2-2+1}{(1+t^2)^2}=\frac{t^4+2t^2+1}{(1+t^2)^2}-\frac{2(t^2+1)}{(1+t^2)^2}+\frac{1}{(1+t^2)^2}=1-\frac{2}{1+t^2}+\frac{1}{(1+t^2)^2}}$$

$$=-4\int\left[1-\frac{2}{1+t^2}+\frac{1}{(1+t^2)^2}\right]\mathrm{d}t\overset{\text{page}}{\underset{17}{=}}-4\left[t-2\,\mathrm{arctg}\,t+\frac{1}{2}\,\mathrm{arctg}\,t+\frac{1}{2}\frac{t}{t^2+1}\right]+c=6\,\mathrm{arctg}\,t-4t-\frac{2t}{t^2+1}+c=$$

$$=6\,\mathrm{arctg}\sqrt{\frac{1-x}{1+x}}-4\sqrt{\frac{1-x}{1+x}}-2\sqrt{\frac{1-x}{1+x}}\frac{1+x}{2}+c=6\,\mathrm{arctg}\sqrt{\frac{1-x}{1+x}}-4\sqrt{\frac{1-x}{1+x}}-\sqrt{1-x^2}+c,\quad\text{for }\ x\in(-1;1\rangle.$$

$$\int\sqrt{\left(\frac{1+x}{1-x}\right)^3}\,\mathrm{d}x=\boxed{\begin{array}{l} t=\sqrt{\frac{1+x}{1-x}},\ t^2=\frac{1+x}{1-x}\implies 1+x=t^2-xt^2\implies x=\frac{t^2-1}{1+t^2},\ x\in\langle -1;1) \\ t^2+1=\frac{2}{1-x},\quad \mathrm{d}x=\frac{2t(1+t^2)-2t(t^2-1)}{(1+t^2)^2}\,\mathrm{d}t=\frac{4t\,\mathrm{d}t}{(1+t^2)^2},\quad t\in\langle 0;\infty)\end{array}}=\int\frac{4t^4\,\mathrm{d}t}{(1+t^2)^2}=$$

$$\boxed{\frac{t^4}{(1+t^2)^2}=\frac{t^4+2t^2+1-2t^2-2+1}{(1+t^2)^2}=\frac{t^4+2t^2+1}{(1+t^2)^2}-\frac{2(t^2+1)}{(1+t^2)^2}+\frac{1}{(1+t^2)^2}=1-\frac{2}{1+t^2}+\frac{1}{(1+t^2)^2}}$$

$$=4\int\left[1-\frac{2}{1+t^2}+\frac{1}{(1+t^2)^2}\right]\mathrm{d}t=4\left[t-2\,\mathrm{arctg}\,t+\frac{1}{2}\,\mathrm{arctg}\,t+\frac{1}{2}\frac{t}{t^2+1}\right]+c=4t-6\,\mathrm{arctg}\,t+\frac{2t}{t^2+1}+c=$$

$$=4\sqrt{\frac{1+x}{1-x}}-6\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}}+2\sqrt{\frac{1+x}{1-x}}\frac{1-x}{2}+c=4\sqrt{\frac{1+x}{1-x}}-6\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}}+\sqrt{1-x^2}+c,\quad\text{for }\ x\in\langle -1;1).$$

$$\int\sqrt{\left(\frac{1+x}{1-x}\right)^3}\,\mathrm{d}x=\boxed{\begin{array}{c} x=-t \\ \mathrm{d}x=-\mathrm{d}t\end{array}}=-\int\sqrt{\left(\frac{1-t}{1+t}\right)^3}\,\mathrm{d}t=-\left[6\,\mathrm{arctg}\sqrt{\frac{1-t}{1+t}}-4\sqrt{\frac{1-t}{1+t}}-\sqrt{1-t^2}\right]+c=$$

$$=-6\,\mathrm{arctg}\sqrt{\frac{1+x}{1-x}}+4\sqrt{\frac{1+x}{1-x}}+\sqrt{1-x^2}+c,\qquad\text{for }\ x\in\langle -1;1).$$

$$\int x^2\ln\sqrt{1-x}\,\mathrm{d}x=\int x^2\ln(1-x)^{\frac{1}{2}}\,\mathrm{d}x=\int\frac{x^2}{2}\ln(1-x)\,\mathrm{d}x=\boxed{\begin{array}{ll} u=\ln(1-x) & \Rightarrow\ u'=\frac{-1}{1-x}=\frac{1}{x-1} \\ v'=\frac{x^2}{2} & \Rightarrow\ v=\frac{x^3}{2\cdot 3}=\frac{x^3}{6}\end{array}}=$$

$$=\frac{x^3\ln(1-x)}{6}-\int\frac{x^3}{6}\frac{\mathrm{d}x}{x-1}=\frac{x^3\ln(1-x)}{6}-\frac{1}{6}\int\frac{(x^3-x^2)+(x^2-x)+(x-1)+1}{x-1}\,\mathrm{d}x=$$

$$=\frac{x^3\ln(1-x)}{6}-\frac{1}{6}\int\left[x^2+x+1+\frac{1}{x-1}\right]\mathrm{d}x=\frac{x^3\ln(1-x)}{6}-\frac{1}{6}\left[\frac{x^2}{3}+\frac{x^2}{2}+x+\ln|x-1|\right]+c=$$

$$=\frac{x^3\ln(1-x)}{6}-\frac{x^3}{18}-\frac{x^2}{12}-\frac{x}{6}-\frac{\ln(1-x)}{6}+c=\frac{x^3-1}{6}\ln(1-x)-\frac{x^3}{18}-\frac{x^2}{12}-\frac{x}{6}+c,\qquad\text{for }\ x<-1.$$

$$\int \frac{dx}{x^6(1+x^2)} = \int \left[\frac{1}{x^2} - \frac{1}{x^4} + \frac{1}{x^6} - \frac{1}{1+x^2}\right] dx = \int \left[x^{-2} - x^{-4} + x^{-6} - \frac{1}{1+x^2}\right] dx = \frac{x^{-1}}{-1} - \frac{x^{-3}}{-3} + \frac{x^{-5}}{-5} - \operatorname{arctg} x + c =$$

$$\boxed{x^2 = t \implies \frac{1}{x^6(1+x^2)} = \frac{1}{t^3(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{1+t} = \frac{A}{x^2} + \frac{B}{x^4} + \frac{C}{x^6} + \frac{D}{1+x^2} \implies A=1,\ B=-1,\ C=1,\ D=-1}$$

$$= -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} - \operatorname{arctg} x + c, \qquad \text{for } x \in R - \{0\}.$$

$$\int \sqrt{\frac{1-x}{1+x}}\, dx = \boxed{\begin{array}{c} t = \sqrt{\dfrac{1-x}{1+x}},\ t^2 = \dfrac{1-x}{1+x} \implies 1-x = t^2 + xt^2 \implies x = \dfrac{1-t^2}{1+t^2},\ x \in (-1;1\rangle \\ t^2+1 = \dfrac{1-x}{1+x} + 1 = \dfrac{2}{1+x}, \quad dx = \dfrac{-2t(1+t^2) - 2t(1-t^2)}{(1+t^2)^2}\, dt = \dfrac{-4t\, dt}{(1+t^2)^2}, \quad t \in \langle 0;\infty) \end{array}} = \int \frac{-4t^2\, dt}{(1+t^2)^2} =$$

$$= -4 \int \frac{t^2+1-1}{(1+t^2)^2}\, dt = -4 \int \left[\frac{1}{1+t^2} - \frac{1}{(1+t^2)^2}\right] dt = -4\left[\operatorname{arctg} t - \frac{1}{2}\operatorname{arctg} t - \frac{1}{2}\frac{t}{t^2+1}\right] + c$$ $\frac{\text{page}}{17}$

$$= -2\operatorname{arctg} t + \frac{2t}{t^2+1} + c = 2\sqrt{\frac{1-x}{1+x}}\frac{1+x}{2} - 2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}} + c = \sqrt{1-x^2} - 2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}} + c, \quad \text{for } x \in (-1;1\rangle.$$

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = \boxed{\begin{array}{c} t = \sqrt{\dfrac{1+x}{1-x}},\ t^2 = \dfrac{1+x}{1-x} \implies 1+x = t^2 - xt^2 \implies x = \dfrac{t^2-1}{1+t^2},\ x \in \langle -1;1) \\ t^2+1 = \dfrac{1+x}{1-x} + 1 = \dfrac{2}{1-x}, \quad dx = \dfrac{2t(1+t^2) - 2t(t^2-1)}{(1+t^2)^2}\, dt = \dfrac{4t\, dt}{(1+t^2)^2}, \quad t \in \langle 0;\infty) \end{array}} = \int \frac{4t^2\, dt}{(1+t^2)^2} =$$

$$= 4\int \frac{t^2+1-1\, dt}{(1+t^2)^2} = 4\int \left[\frac{1}{1+t^2}1\frac{1}{(1+t^2)^2}\right] dt = 4\left[\operatorname{arctg} t - \frac{1}{2}\operatorname{arctg} t - \frac{1}{2}\frac{t}{t^2+1}\right] + c = 2\operatorname{arctg} t - \frac{2t}{t^2+1} + c =$$

$$= 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - 2\sqrt{\frac{1+x}{1-x}}\frac{1-x}{2} + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c, \qquad \text{for } x \in \langle -1;1).$$

$$\int \sqrt{\frac{1+x}{1-x}}\, dx = \boxed{\begin{array}{c} x = -t \\ dx = -\, dt \end{array}} = -\int \sqrt{\frac{1-t}{1+t}}\, dt = -\left[\sqrt{1-t^2} - 2\operatorname{arctg}\sqrt{\frac{1-t}{1+t}}\right] + c = 2\operatorname{arctg}\sqrt{\frac{1+x}{1-x}} - \sqrt{1-x^2} + c, \quad \text{for } x \in \langle -1;1).$$

$$\int \sqrt{\frac{x-1}{x+1}}\, dx = \int \frac{4t^2\, dt}{(t-1)^2(t+1)^2} = \boxed{\frac{4t^2}{(t^2-1)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2} \implies A=1,\ B=1,\ C=-1,\ D=1} =$$

$$\boxed{\begin{array}{c} t = \sqrt{\dfrac{x-1}{x+1}},\ t^2 = \dfrac{x-1}{x+1} \implies x-1 = xt^2 + t^2 \implies x = \dfrac{1+t^2}{1-t^2},\ x \in (-\infty;-1) \cup \langle 1;\infty),\ \text{ i.e. } x \in R - \langle -1;1) \\ 1-t^2 = 1 - \dfrac{x-1}{x+1} = \dfrac{2}{x+1} \implies \dfrac{1}{t^2-1} = -\dfrac{x+1}{2}, \quad dx = \dfrac{2t(1-t^2) - (-2t)(1+t^2)}{(1-t^2)^2}\, dt = \dfrac{4t\, dt}{(t^2-1)^2}, \quad t \in \langle 0;\infty) - \{1\} \end{array}}$$

$$= \int \left[\frac{1}{t-1} + \frac{1}{(t-1)^2} - \frac{1}{t+1} + \frac{1}{(t+1)^2}\right] dt = \ln|t-1| + \int (t-1)^{-2}\, dt - \ln|t+1| + \int (t+1)^{-2}\, dt =$$

$$= \ln\left|\frac{t-1}{t+1}\right| + \frac{(t-1)^{-1}}{-1} + \frac{(t+1)^{-1}}{-1} + c = \ln\left|\frac{t-1}{t+1}\right| - \frac{1}{t-1} - \frac{1}{t+1} + c = \ln|t-1| - \ln|t+1| - \frac{2t}{t^2-1} + c =$$

$$= \ln\left|\sqrt{\frac{x-1}{x+1}} - 1\right| - \ln\left|\sqrt{\frac{x-1}{x+1}} + 1\right| + (x+1)\sqrt{\frac{x-1}{x+1}} + c, \qquad \text{for } x \in (-\infty;-1) \cup \langle 1;\infty).$$

$$\int\sqrt{\frac{x+1}{x-1}}\,dx=\int\frac{-4t^2\,dt}{(t-1)^2\,(t+1)^2}=\boxed{\frac{4t^2}{(t^2-1)^2}=\frac{A}{t-1}+\frac{B}{(t-1)^2}+\frac{C}{t+1}+\frac{D}{(t+1)^2}\implies A=1,\ B=1,\ C=-1,\ D=1}=$$

$$\boxed{\begin{array}{l} t=\sqrt{\frac{x+1}{x-1}},\ t^2=\frac{x+1}{x-1}\implies x+1=xt^2-t^2\implies x=\frac{t^2+1}{t^2-1},\ x\in(-\infty;-1)\cup\langle 1;\infty),\ \text{i.e. } x\in R-\langle -1;1) \\ t^2-1=\frac{x+1}{x-1}-1=\frac{2}{x-1}\implies\frac{1}{t^2-1}=\frac{x-1}{2},\quad dx=\frac{2t(t^2-1)-2t(t^2+1)}{(t^2-1)^2}\,dt=\frac{-4t\,dt}{(t^2-1)^2},\quad t\in\langle 0;\infty)-\{1\}\end{array}}$$

$$=-\int\left[\frac{1}{t-1}+\frac{1}{(t-1)^2}-\frac{1}{t+1}+\frac{1}{(t+1)^2}\right]dt=-\ln|t-1|-\int(t-1)^{-2}\,dt+\ln|t+1|-\int(t+1)^{-2}\,dt=$$

$$=\ln\left|\frac{t+1}{t-1}\right|-\frac{(t-1)^{-1}}{-1}-\frac{(t+1)^{-1}}{-1}+c=\ln\left|\frac{t+1}{t-1}\right|+\frac{1}{t-1}+\frac{1}{t+1}+c=\ln|t+1|-\ln|t-1|+\frac{2t}{t^2-1}+c=$$

$$=\ln\left|\sqrt{\frac{x+1}{x-1}}+1\right|-\ln\left|\sqrt{\frac{x+1}{x-1}}-1\right|+(x-1)\sqrt{\frac{x+1}{x-1}}+c,\qquad\text{for }\ x\in(-\infty;-1)\cup\langle 1;\infty).$$

$$\int\frac{dx}{x\sqrt{1-x^3+x^6}}=\boxed{x^6-x^3+1=\left[x^3-\frac{1}{2}\right]^2+1-\frac{1}{4}>0}=\frac{1}{3}\int\frac{3x^2\,dx}{x^3\sqrt{1-x^3+x^6}}=\boxed{\begin{array}{c}t=x^3\\3x^2\,dx=dt\end{array}}=\frac{1}{3}\int\frac{dt}{t\sqrt{1-t+t^2}}=$$

$$\boxed{t=\frac{1}{u}\implies dt=-\frac{du}{u^2},\ \sqrt{1-t+t^2}=\sqrt{1-\frac{1}{u}+\frac{1}{u^2}}=\frac{\sqrt{u^2-u+1}}{u},\ \sqrt{u^2-u+1}=\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}=\frac{\sqrt{1-t+t^2}}{t}}$$

$$=\frac{1}{3}\int u\frac{u}{\sqrt{u^2-u+1}}\frac{-du}{u^2}=-\frac{1}{3}\int\frac{du}{\sqrt{\left(u-\frac{1}{2}\right)^2+1-\frac{1}{4}}}=\boxed{\begin{array}{c}v=u-1/2\\dv=du\end{array}}=-\frac{1}{3}\int\frac{dv}{\sqrt{v^2+\frac{3}{4}}}=$$

$$=-\frac{1}{3}\ln\left|v+\sqrt{v^2+\frac{3}{4}}\right|+c_1=c_1-\frac{1}{3}\ln\left|u-\frac{1}{2}+\sqrt{u^2-u+1}\right|=c_1-\frac{1}{3}\ln\left|\frac{1}{t}-\frac{1}{2}+\frac{\sqrt{1-t+t^2}}{t}\right|=$$

$$=c_1-\frac{1}{3}\ln\left|\frac{2-t+2\sqrt{1-t+t^2}}{2t}\right|=c_1-\frac{1}{3}\left[\ln\left|2-t+2\sqrt{1-t+t^2}\right|-\ln|t|-\ln 2\right]=$$

$$=c_1-\frac{1}{3}\ln\left|2-x^3+2\sqrt{1-x^3+x^6}\right|+\frac{1}{3}\ln|x^3|+\frac{1}{3}\ln 2=c-\frac{1}{3}\ln\left|2-x^3+2\sqrt{1-x^3+x^6}\right|+\ln x,\ \text{ for } x\in R-\{0\}.$$

$$\int\frac{dx}{x\sqrt{1+x^3+x^6}}=\boxed{x^6+x^3+1=\left[x^3+\frac{1}{2}\right]^2+1-\frac{1}{4}>0}=\frac{1}{3}\int\frac{3x^2\,dx}{x^3\sqrt{1+x^3+x^6}}=\boxed{\begin{array}{c}t=x^3\\3x^2\,dx=dt\end{array}}=\frac{1}{3}\int\frac{dt}{t\sqrt{1+t+t^2}}=$$

$$\boxed{t=\frac{1}{u}\implies dt=-\frac{du}{u^2},\ \sqrt{1+t+t^2}=\sqrt{1+\frac{1}{u}+\frac{1}{u^2}}=\frac{\sqrt{u^2+u+1}}{u},\ \sqrt{u^2+u+1}=\sqrt{\frac{1}{t^2}+\frac{1}{t}+1}=\frac{\sqrt{1+t+t^2}}{t}}$$

$$=\frac{1}{3}\int u\frac{u}{\sqrt{u^2+u+1}}\frac{-du}{u^2}=-\frac{1}{3}\int\frac{du}{\sqrt{\left(u+\frac{1}{2}\right)^2+1-\frac{1}{4}}}=\boxed{\begin{array}{c}v=u+1/2\\dv=du\end{array}}=-\frac{1}{3}\int\frac{dv}{\sqrt{v^2+\frac{3}{4}}}=$$

$$=-\frac{1}{3}\ln\left|v+\sqrt{v^2+\frac{3}{4}}\right|+c_1=c_1-\frac{1}{3}\ln\left|u+\frac{1}{2}+\sqrt{u^2+u+1}\right|=c_1-\frac{1}{3}\ln\left|\frac{1}{t}+\frac{1}{2}+\frac{\sqrt{1+t+t^2}}{t}\right|=$$

$$=c_1-\frac{1}{3}\ln\left|\frac{2+t+2\sqrt{1+t+t^2}}{2t}\right|=c_1-\frac{1}{3}\left[\ln\left|2+t+2\sqrt{1+t+t^2}\right|-\ln|t|-\ln 2\right]=$$

$$=c_1-\frac{1}{3}\ln\left|2+x^3+2\sqrt{1+x^3+x^6}\right|+\frac{1}{3}\ln|x^3|+\frac{1}{3}\ln 2=c-\frac{1}{3}\ln\left|2+x^3+2\sqrt{1+x^3+x^6}\right|+\ln x,\ \text{ for } x\in R-\{0\}.$$

$$\int \ln\left(\sqrt{1+x}+\sqrt{1-x}\right) dx = x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) - \int x\left[\frac{1}{2x}-\frac{1}{2x\sqrt{1-x^2}}\right] dx = x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) - \int\frac{dx}{2}+\int\frac{dx}{2\sqrt{1-x^2}} =$$

$$v'=1 \quad\Rightarrow\quad v=x$$

$$u=\ln\left(\sqrt{1+x}+\sqrt{1-x}\right) \;\Rightarrow\; u'=\frac{\frac{1}{2}\frac{1}{\sqrt{1+x}}-\frac{1}{2}\frac{1}{\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}}=\frac{\frac{\sqrt{1-x}-\sqrt{1+x}}{2\sqrt{1+x}\sqrt{1-x}}}{\sqrt{1+x}+\sqrt{1-x}}=\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1+x}+\sqrt{1-x}}\frac{1}{2\sqrt{1-x^2}}=$$

$$=\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1+x}+\sqrt{1-x}}\frac{1}{2\sqrt{1-x^2}}\frac{\sqrt{1-x}-\sqrt{1+x}}{\sqrt{1-x}-\sqrt{1+x}}=\frac{\left(\sqrt{1-x}-\sqrt{1+x}\right)^2}{1-x-(1+x)}\frac{1}{2\sqrt{1-x^2}}=$$

$$=\frac{1-x-2\sqrt{1-x}\sqrt{1+x}+1+x}{-2x\cdot 2\sqrt{1-x^2}}=\frac{2-2\sqrt{1-x^2}}{-2x\cdot 2\sqrt{1-x^2}}=\frac{\sqrt{1-x^2}-1}{2x\sqrt{1-x^2}}=\frac{1}{2x}-\frac{1}{2x\sqrt{1-x^2}}$$

$$= x\ln\left(\sqrt{1+x}+\sqrt{1-x}\right)-\frac{x}{2}+\frac{\arcsin x}{2}+c, \quad \text{for } \; x\in\langle -1\,;\,1\rangle.$$

$$\int\frac{1-x}{x-\sqrt{x-x^2}}\,dx=\int\frac{t^2}{1+t^2}\frac{1+t^2}{1-t}\frac{-2t\,dt}{(1+t^2)^2}=\int\frac{2t^3\,dt}{(t-1)\,(1+t^2)^2}=\int\left[\frac{\frac{1}{2}}{t-1}+\frac{\frac{3}{2}-\frac{t}{2}}{1+t^2}+\frac{t-1}{(1+t^2)^2}\right]dt=$$

3-rd Euler: $t=\sqrt{\dfrac{1-x}{x}} \implies t^2=\dfrac{1-x}{x} \implies t^2x=1-x \implies x=\dfrac{1}{1+t^2} \implies dx=\dfrac{-1\cdot 2t}{(1+t^2)^2}\,dt=\dfrac{-2t\,dt}{(1+t^2)^2}$

$$x-x^2=x(1-x)>0 \implies x\in(0\,;\,1) \implies t\in(0\,;\,\infty)\,, \qquad 1-x=1-\frac{1}{1+t^2}=\frac{1+t^2-1}{1+t^2}=\frac{t^2}{1+t^2}$$

$$x-\sqrt{x(1-x)}=\frac{1}{1+t^2}-\sqrt{\frac{1}{1+t^2}\frac{t^2}{1+t^2}}=\frac{1}{1+t^2}-\frac{t}{1+t^2}=\frac{1-t}{1+t^2}, \qquad t=\sqrt{\frac{1-x}{x}}=\sqrt{\frac{x(1-x)}{x^2}}=\frac{\sqrt{x-x^2}}{x}$$

$$\frac{t}{1+t^2}=\frac{1}{1+t^2}t=x\frac{\sqrt{x-x^2}}{x}=\sqrt{x-x^2}, \qquad \frac{1+t}{1+t^2}=\frac{1}{1+t^2}+\frac{t}{1+t^2}=x+\sqrt{x-x^2}$$

$$\frac{2t^3}{(t-1)\,(1+t^2)^2}=\frac{A}{t-1}+\frac{Bt+C}{1+t^2}+\frac{Dt+E}{(1+t^2)^2} \implies A=\frac{1}{2},\; B=-\frac{1}{2},\; C=\frac{3}{2},\; D=1,\; E=-1$$

$$=\frac{1}{2}\int\frac{dt}{t-1}+\frac{3}{2}\int\frac{dt}{1+t^2}-\frac{1}{2}\int\frac{t\,dt}{1+t^2}+\int\frac{t\,dt}{(1+t^2)^2}-\int\frac{dt}{(1+t^2)^2}=\boxed{\begin{array}{c}1+t^2=u\\ 2t\,dt=du\end{array}}=$$

$$=\frac{1}{2}\ln|t-1|+\frac{3}{2}\operatorname{arctg} t-\frac{1}{4}\int\frac{du}{u}+\frac{1}{2}\int\frac{du}{u^2}-\int\frac{dt}{(1+t^2)^2}\;\overset{\text{page}}{\underset{17}{=}}\;\boxed{\int\frac{dx}{(x^2+1)^2}=\frac{1}{2}\operatorname{arctg} x+\frac{1}{2}\frac{x}{x^2+1}}=$$

$$=\frac{1}{2}\ln|t-1|+\frac{3}{2}\operatorname{arctg} t-\frac{1}{4}\ln|u|+\frac{1}{2}\frac{u^{-1}}{-1}-\frac{1}{2}\operatorname{arctg} t+\frac{t}{2(t^2+1)}+c=\frac{1}{4}\ln(t-1)^2+\operatorname{arctg} t-\frac{1}{4}\ln u-\frac{1}{2u}+\frac{t}{2(t^2+1)}+c=$$

$$=\frac{1}{4}\ln(t-1)^2+\operatorname{arctg} t-\frac{1}{4}\ln\left(1+t^2\right)-\frac{1}{2(1+t^2)}-\frac{t}{2(1+t^2)}+c=\frac{1}{4}\ln\frac{t^2-2t+1}{1+t^2}+\operatorname{arctg} t-\frac{1+t}{2(1+t^2)}+c=$$

$$=\frac{1}{4}\ln\left(1-\frac{2t}{1+t^2}\right)+\operatorname{arctg} t-\frac{1+t}{2(1+t^2)}+c=\frac{1}{4}\ln\left(1-2\sqrt{x-x^2}\right)+\operatorname{arctg}\frac{\sqrt{x-x^2}}{x}-\frac{x+\sqrt{x-x^2}}{2}+c,$$

for $\; x\in(0\,;\,1)$.

$$\int \frac{1-x}{x-\sqrt{x^2-x}}\,dx = \int \frac{(1-x)\left(x+\sqrt{x^2-x}\right)}{x^2-(x^2-x)}\,dx = \int \frac{x-x^2+\sqrt{x^2-x}-x\sqrt{x^2-x}}{x}\,dx = \int (1-x)\,dx + \int \frac{\sqrt{x^2-x}}{x}\,dx - \int \sqrt{x^2-x}\,dx =$$

$$= x-\frac{x^2}{2} + \int \frac{\sqrt{x^2-x}}{x}\,dx - \int \sqrt{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}}\,dx = \boxed{\begin{array}{c} x-1/2=t \\ dx=dt \end{array}} = x-\frac{x^2}{2} + \int \frac{\sqrt{x^2-x}}{x}\,dx - \int \sqrt{t^2-\frac{1}{4}}\,dt \stackrel{\text{page}}{\underset{49,56}{=}}$$

$$\boxed{\int \frac{\sqrt{x^2-x}}{x}\,dx = \frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x}, \qquad \int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|}$$

$$= x-\frac{x^2}{2} + \left[\frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x}\right] - \left[\frac{t\sqrt{t^2-1/4}}{2} - \frac{1}{2}\cdot\frac{1}{4}\ln\left|t+\sqrt{t^2-1/4}\right|\right] + c_1 =$$

$$= x-\frac{x^2}{2} + \frac{1}{2}\ln\left(2x-1-2\sqrt{x^2-x}\right)+\sqrt{x^2-x} - \frac{2x-1}{4}\sqrt{x^2-x} + \frac{1}{8}\ln\left|x-\frac{1}{2}+\sqrt{x^2-x}\right| + c_1 =$$

$$= x-\frac{x^2}{2} + \frac{4}{8}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + \frac{1}{8}\ln\left|\frac{2x-1+2\sqrt{x^2-x}}{2}\right| + c_1 =$$

$$= x-\frac{x^2}{2} + \frac{1+3}{4}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + \frac{1}{8}\ln\left|2x-1+2\sqrt{x^2-x}\right| - \frac{1}{8}\ln 2 + c_1 =$$

$$= x-\frac{x^2}{2} + \frac{1}{8}\ln\left|\left(2x-1-2\sqrt{x^2-x}\right)\left(2x-1+2\sqrt{x^2-x}\right)\right| + \frac{3}{8}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + c =$$

$$= x-\frac{x^2}{2} + \frac{1}{8}\ln\left|(2x-1)^2-4(x^2-x)\right| + \frac{3}{8}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + c =$$

$$= x-\frac{x^2}{2} + \frac{1}{8}\ln\left|4x^2-4x+1-4x^2+4x\right| + \frac{3}{8}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + c =$$

$$= x-\frac{x^2}{2} + \frac{1}{8}\ln 1 + \frac{3}{8}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + c =$$

$$= x-\frac{x^2}{2} + \frac{3}{8}\ln\left(2x-1-2\sqrt{x^2-x}\right) + \frac{5-2x}{4}\sqrt{x^2-x} + c, \qquad \text{for } x\in(-\infty;0)\cup(1;\infty).$$

$$\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,dx = \int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,\frac{1+\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\,dx = \int \frac{1+2\sqrt{1-x^2}+1-x^2}{1-(1-x^2)}\,dx = \int \frac{2-x^2+2\sqrt{1-x^2}}{x^2}\,dx =$$

$$= 2\int x^{-2}\,dx - \int dx + 2\int \frac{\sqrt{1-x^2}}{x^2}\,dx \stackrel{\text{page}}{\underset{57}{=}} 2\frac{x^{-1}}{-1} - x + 2\left[-\frac{\sqrt{1-x^2}}{x} - \arcsin x\right] + c =$$

$$= -\frac{2}{x} - x - \frac{2\sqrt{1-x^2}}{x} - 2\arcsin x + c = c - \frac{2+2\sqrt{1-x^2}}{x} - 2\arcsin x - x, \qquad \text{for } x\in\langle -1;1\rangle - \{0\}.$$

$$\int \sqrt{\frac{x}{1-x\sqrt{x}}}\,dx = \boxed{\begin{array}{c} \sqrt{x}=t,\ \ x=t^2,\ \ dx=2t\,dt \\ x\in\langle 0;1),\quad t\in\langle 0;1) \end{array}} = \int \sqrt{\frac{t^2}{1-t^3}}\,2t\,dt = \int \frac{2t^2\,dt}{\sqrt{1-t^3}} = \boxed{\begin{array}{c} 1-t^3=u \\ -3t^2\,dt=du \end{array}} = -\frac{2}{3}\int \frac{du}{\sqrt{u}} =$$

$$\boxed{x\geq 0,\quad 1-x\sqrt{x}>0 \implies x\geq 0,\quad 1>x\sqrt{x}=\sqrt{x^3} \implies x\geq 0,\quad 1>x \implies x\in\langle 0;1)}$$

$$= -\frac{2}{3}\int u^{-\frac{1}{2}}\,du = -\frac{2}{3}\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = c - \frac{4}{3}\sqrt{u} = c - \frac{4}{3}\sqrt{1-t^3} = c - \frac{4}{3}\sqrt{1-x\sqrt{x}}, \qquad \text{for } x\in\langle 0;1).$$

$$\int \frac{dx}{\sqrt{e^{2x}+e^x+1}} = \boxed{e^x = t>0,\ \ x=\ln t \in (-\infty;\infty),\ \ dx=\frac{dt}{t}} = \int \frac{dt}{t\sqrt{t^2+t+1}} = -\int \frac{\frac{du}{u^2}}{\frac{1}{u}\frac{\sqrt{1+u+u^2}}{u}} = -\int\frac{du}{\sqrt{1+u+u^2}} =$$

$$\boxed{t=\frac1u \Longrightarrow dt=-\frac{du}{u^2},\quad \sqrt{t^2+t+1}=\sqrt{\frac1{u^2}+\frac1u+1}=\frac{\sqrt{1+u+u^2}}{u},\quad \sqrt{1+u+u^2}=\sqrt{1+\frac1t+\frac1{t^2}}=\frac{\sqrt{t^2+t+1}}{t}}$$

$$= -\int \frac{du}{\sqrt{\left(u+\frac12\right)^2+1-\frac14}} = \boxed{u+\frac12=v,\ \ du=dv} = -\int\frac{dv}{\sqrt{v^2+\frac34}} = -\ln\left[v+\sqrt{v^2+\frac34}\right]+c_1=$$

$$=c_1-\ln\left[u+\frac12+\sqrt{1+u+u^2}\right] = c_1-\ln\left[\frac1t+\frac12+\frac{\sqrt{t^2+t+1}}{t}\right] = c_1-\ln\left[\frac{2+t+2\sqrt{t^2+t+1}}{2t}\right]=$$

$$=c_1-\ln\left(2+t+2\sqrt{t^2+t+1}\right)+\ln2+\ln t = c-\ln\left(2+e^x+2\sqrt{e^{2x}+e^x+1}\right)+x, \qquad \text{for } x\in R.$$

$$\int\sqrt{\frac{1-e^x}{1+e^x}}\,dx = \boxed{\begin{array}{c} e^x=t \Longrightarrow x=\ln t,\ dx=\frac{dt}{t}\\ x\in(-\infty;0\rangle \Longrightarrow t\in(0;1\rangle\end{array}} = \int\sqrt{\frac{1-t}{1+t}}\frac{dt}{t} = \int u\frac{1+u^2}{1-u^2}\frac{-4u\,du}{(1+u^2)^2} = \int\frac{-4u^2\,du}{(1-u^2)(1+u^2)}=$$

$$\boxed{\begin{array}{l} u=\sqrt{\frac{1-t}{1+t}},\ u^2=\frac{1-t}{1+t} \Longrightarrow u^2+u^2t=1-t,\ t=\frac{1-u^2}{1+u^2},\ dt=\frac{-2u(1+u^2)-2u(1-u^2)}{(1+u^2)^2}du=\frac{-4u\,du}{(1+u^2)^2},\ u\in\langle0;1)\\ 1\pm u=1\pm\sqrt{\frac{1-t}{1+t}}=\frac{\sqrt{1+t}\pm\sqrt{1-t}}{\sqrt{1+t}} \Longrightarrow \frac{1-u}{1+u}=\frac{\sqrt{1+e^x}-\sqrt{1-e^x}}{\sqrt{1+e^x}+\sqrt{1-e^x}}=\frac{1+e^x-2\sqrt{1-e^{2x}}+1-e^x}{(1+e^x)-(1-e^x)}=\frac{1-\sqrt{1-e^{2x}}}{e^x}\\ \hline \frac{-4u^2}{(1-u^2)(1+u^2)}=\frac{2}{1+u^2}-\frac{2}{1-u^2}=\frac{2}{1+u^2}-\frac{2}{(u-1)(u+1)}=\frac{2}{1+u^2}-\left[\frac{1}{u+1}-\frac{1}{u-1}\right]=\frac{2}{1+u^2}-\frac{1}{u+1}+\frac{1}{u-1}\end{array}}$$

$$=\int\left[\frac2{1+u^2}-\frac1{u+1}+\frac1{u-1}\right]du = 2\,\mathrm{arctg}\,u-\ln|u+1|+\ln|u-1|+c=2\,\mathrm{arctg}\,u+\ln\left|\frac{u-1}{u+1}\right|+c=$$

$$=2\,\mathrm{arctg}\,u+\ln\frac{1-u}{1+u}+c=2\,\mathrm{arctg}\sqrt{\frac{1-e^x}{1+e^x}}+\ln\frac{1-\sqrt{1-e^{2x}}}{e^x}+c=$$

$$=2\,\mathrm{arctg}\sqrt{\frac{1-e^x}{1+e^x}}+\ln\left(1-\sqrt{1-e^{2x}}\right)-\ln e^x+c = 2\,\mathrm{arctg}\sqrt{\frac{1-e^x}{1+e^x}}+\ln\left(1-\sqrt{1-e^{2x}}\right)-x+c, \qquad \text{for } x\in(-\infty;0\rangle.$$

$$\int\frac{1-x}{x\sqrt{x-x^2}}\,dx = \int\frac{t^2}{1+t^2}\,\frac{1+t^2}{1}\,\frac{1+t^2}{t}\,\frac{-2t\,dt}{(1+t^2)^2} = -2\int\frac{t^2\,dt}{1+t^2} = -2\int\frac{1+t^2-1}{1+t^2}\,dt = -2\int\left[1-\frac1{1+t^2}\right]dt=$$

$$\boxed{\begin{array}{l} \text{3-rd Euler:}\quad t=\sqrt{\frac{1-x}{x}} \Longrightarrow t^2=\frac{1-x}{x} \Longrightarrow x=\frac1{1+t^2},\ 1-x=\frac{t^2}{1+t^2} \Longrightarrow dx=\frac{-1\cdot2t}{(1+t^2)^2}dt=\frac{-2t\,dt}{(1+t^2)^2}\\ x-x^2=x(1-x)>0 \Longrightarrow x\in(0;1) \Longrightarrow t\in(1;\infty),\quad \sqrt{x(1-x)}=\sqrt{\frac1{1+t^2}\,\frac{t^2}{1+t^2}}=\frac{t}{1+t^2}\end{array}}$$

$$=-2(t-\mathrm{arctg}\,t)+c=2\,\mathrm{arctg}\sqrt{\frac{1-x}{x}}-2\sqrt{\frac{1-x}{x}}+c, \qquad \text{for } x\in(0;1).$$

$$\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x} = \boxed{\sin^2 x = \frac{1-\cos 2x}{2},\ \cos^2 x = \frac{1+\cos 2x}{2}} = \int \frac{2\,dx}{a^2(1+\cos 2x) + b^2(1-\cos 2x)} =$$

$$= \int \frac{2\,dx}{(a^2+b^2)+(a^2-b^2)\cos 2x} = \boxed{\begin{matrix} 2x = t \\ 2x\,dx = dt \end{matrix}} = \int \frac{dt}{(a^2+b^2)+(a^2-b^2)\cos t} = \int \frac{u^2+1}{2b^2u^2+2a^2}\frac{2\,du}{u^2+1} = \int \frac{du}{b^2u^2+a^2} =$$

$$\boxed{u = \operatorname{tg}\frac{t}{2},\ dt = \frac{2\,du}{u^2+1},\ \cos t = \frac{1-u^2}{u^2+1} \implies (a^2+b^2)+(a^2-b^2)\cos t = (a^2+b^2)\frac{u^2+1}{u^2+1} + (a^2-b^2)\frac{1-u^2}{u^2+1} = \frac{2b^2u^2+2a^2}{u^2+1}}$$

$$= \frac{1}{b^2}\int \frac{du}{u^2+\frac{a^2}{b^2}} = \frac{1}{b^2}\frac{b}{a}\operatorname{arctg}\frac{bu}{a} + c = \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg}\frac{t}{2}\right) + c = \frac{1}{ab}\operatorname{arctg}\left(\frac{b}{a}\operatorname{tg} x\right) + c, \qquad \text{for } x \in R,\ a, b \in R - \{0\}.$$

$$\int \frac{dx}{a^2\cos^2 x - b^2\sin^2 x} = \boxed{\sin^2 x = \frac{1-\cos 2x}{2},\ \cos^2 x = \frac{1+\cos 2x}{2}} = \int \frac{2\,dx}{a^2(1+\cos 2x) - b^2(1-\cos 2x)} =$$

$$= \int \frac{2\,dx}{(a^2-b^2)+(a^2+b^2)\cos 2x} = \boxed{\begin{matrix} 2x = t \\ 2x\,dx = dt \end{matrix}} = \int \frac{dt}{(a^2-b^2)+(a^2+b^2)\cos t} = \int \frac{u^2+1}{2a^2-2b^2u^2}\frac{2\,du}{u^2+1} = \int \frac{du}{a^2-b^2u^2} =$$

$$\boxed{u = \operatorname{tg}\frac{t}{2},\ dt = \frac{2\,du}{u^2+1},\ \cos t = \frac{1-u^2}{u^2+1} \implies (a^2-b^2)+(a^2+b^2)\cos t = (a^2-b^2)\frac{u^2+1}{u^2+1} + (a^2+b^2)\frac{1-u^2}{u^2+1} = \frac{2a^2-2b^2u^2}{u^2+1}}$$

$$= -\frac{1}{b^2}\int \frac{du}{u^2-\frac{a^2}{b^2}} = -\frac{1}{b^2}\frac{b}{2a}\ln\left|\frac{u-\frac{a}{b}}{u+\frac{a}{b}}\right| + c = -\frac{1}{2ab}\ln\left|\frac{bu-a}{bu+a}\right| + c = -\frac{1}{2ab}\ln\left|\frac{b\operatorname{tg} x - a}{b\operatorname{tg} x + a}\right| + c, \qquad \text{for } x \in R,\ a, b \in R - \{0\}.$$

$$\int x\operatorname{tg}^2 x\,dx = \int \frac{x\sin^2 x}{\cos^2 x}\,dx = \int \frac{x(1-\cos^2 x)}{\cos^2 x}\,dx = \int \frac{x\,dx}{\cos^2 x} - \int x\,dx = \int \frac{x\,dx}{\cos^2 x} - \frac{x^2}{2} = \boxed{\begin{matrix} u = x & \Rightarrow & u' = 1 \\ v' = \frac{1}{\cos^2 x} & \Rightarrow & v = \operatorname{tg} x = \frac{\sin x}{\cos x} \end{matrix}} =$$

$$= x\operatorname{tg} x - \int \frac{\sin x}{\cos x}\,dx - \frac{x^2}{2} = x\operatorname{tg} x + \int \frac{-\sin x}{\cos x}\,dx - \frac{x^2}{2} = x\operatorname{tg} x + \ln|\cos x| - \frac{x^2}{2} + c, \quad \text{for } x \in R - \left\{(2k+1)\frac{\pi}{2}, k \in Z\right\}.$$

$$\int \frac{dx}{1+\sqrt[3]{x}} = \boxed{\begin{matrix} x \geq 0,\ x = t^3 \geq 0 \\ t = \sqrt[3]{x},\ dx = 3t^2\,dt \end{matrix}} = \int \frac{3t^2\,dt}{1+t} = 3\int \frac{t^2\,dt}{1+t} = 3\int \frac{t^2+t-t-1+1}{1+t}\,dt = 3\int \left[t-1+\frac{1}{1+t}\right]dt =$$

$$= \frac{3t^2}{2} - 3t + 3\ln|1+t| + c = \frac{3t^2}{2} - 3t + 3\ln(1+t) + c = \frac{3\sqrt[3]{x^2}}{2} - 3\sqrt[3]{x} + 3\ln\left(1+\sqrt[3]{x}\right) + c, \qquad \text{for } x \in \langle 0\,;\,\infty)$$

$$\int \frac{dx}{\sqrt[3]{x}+\sqrt[4]{x}} = \int \frac{12t^{11}\,dt}{t^4+t^3} = 12\int \frac{t^8\,dt}{t+1} = 12\int \frac{t^8+t^7-t^7-t^6+t^6+t^5-t^5-t^4+t^4+t^3-t^3-t^2+t^2+t-t-1+1}{t+1}\,dt =$$

$$\boxed{x > 0,\ x = t^{12} > 0,\ t = \sqrt[12]{x},\ dx = 12t^{11}\,dt,\ t^8 = \sqrt[12]{x^8} = \sqrt[3]{x^2},\ t^6 = \sqrt[12]{x^6} = \sqrt{x},\ t^4 = \sqrt[12]{x^4} = \sqrt[3]{x},\ t^3 = \sqrt[12]{x^3} = \sqrt[4]{x},\ t^2 = \sqrt[12]{x^2} = \sqrt[6]{x}}$$

$$= 12\int \left[t^7-t^6+t^5-t^4+t^3-t^2+t-1+\frac{1}{t+1}\right]dt = 12\left[\frac{t^8}{8}-\frac{t^7}{7}+\frac{t^6}{6}-\frac{t^5}{5}+\frac{t^4}{4}-\frac{t^3}{3}+\frac{t^2}{2}-t+\ln|1+t|\right] + c =$$

$$= \frac{3\sqrt[3]{x^2}}{2} - \frac{12\sqrt[12]{x^7}}{7} + 2\sqrt{x} - \frac{12\sqrt[12]{x^5}}{5} + 3\sqrt[3]{x} - 4\sqrt[4]{x} + 6\sqrt[6]{x} - 12\sqrt[12]{x} + 12\ln\left|1+\sqrt[12]{x}\right| + c, \qquad \text{for } x \in (0\,;\,\infty)$$

$$\int \frac{\mathrm{d}x}{\sqrt[n]{x^n+1}} = \int \frac{\sqrt[n]{t^n-1}}{t} \frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n-1)^{n+1}}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{\sqrt[n]{(t^n-1)^n}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{t^n-1} = \cdots \text{ [Partial fractions]}, \quad \text{for } x\in(0\,;\,\infty),\ n=2,3,4,\ldots.$$

$$\boxed{\begin{array}{l} x>0, \quad t=\sqrt[n]{1+\frac{1}{x^n}}=\frac{\sqrt[n]{x^n+1}}{x}>1, \quad t^n=1+\frac{1}{x^n} \implies t^nx^n=x^n+1 \implies x^n=\frac{1}{t^n-1} \implies x=\frac{1}{\sqrt[n]{t^n-1}} \\ \mathrm{d}x=-\frac{1}{n}(t^n-1)^{-\frac{1}{n}-1}nt^{n-1}\,\mathrm{d}t=\frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n-1)^{n+1}}}, \quad x^n+1=\frac{1}{t^n-1}+1=\frac{t^n}{t^n-1} \implies \sqrt[n]{x^n+1}=t\,(t^n-1)^{-\frac{1}{n}}=\frac{t}{\sqrt[n]{t^n-1}} \end{array}}$$

$$\int \frac{\mathrm{d}x}{\sqrt[n]{1-x^n}} = \int \frac{\sqrt[n]{t^n+1}}{t} \frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n+1)^{n+1}}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{\sqrt[n]{(t^n+1)^n}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{t^n+1} = \cdots \text{ [Partial fractions]}, \quad \text{for } x\in(0\,;\,1),\ n=2,3,4,\ldots.$$

$$\boxed{\begin{array}{l} 0<x<1, \quad t=\sqrt[n]{\frac{1}{x^n}-1}=\frac{\sqrt[n]{1-x^n}}{x}>0, \quad t^n=\frac{1}{x^n}-1 \implies t^nx^n=1-x^n \implies x^n=\frac{1}{t^n+1} \implies x=\frac{1}{\sqrt[n]{t^n+1}} \\ \mathrm{d}x=-\frac{1}{n}(t^n+1)^{-\frac{1}{n}-1}nt^{n-1}\,\mathrm{d}t=\frac{-t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(t^n+1)^{n+1}}}, \quad 1-x^n=1-\frac{1}{t^n+1}=\frac{t^n}{t^n+1} \implies \sqrt[n]{1-x^n}=t\,(t^n+1)^{-\frac{1}{n}}=\frac{t}{\sqrt[n]{t^n+1}} \end{array}}$$

$$\int \frac{\mathrm{d}x}{\sqrt[n]{x^n-1}} = \int \frac{\sqrt[n]{1-t^n}}{t} \frac{t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(1-t^n)^{n+1}}} = \int \frac{t^{n-2}\,\mathrm{d}t}{\sqrt[n]{(1-t^n)^n}} = -\int \frac{t^{n-2}\,\mathrm{d}t}{t^n-1} = \cdots \text{ [Partial fractions]}, \quad \text{for } x\in(1\,;\,\infty),\ n=2,3,4,\ldots.$$

$$\boxed{\begin{array}{l} x>1, \quad t=\sqrt[n]{1-\frac{1}{x^n}}=\frac{\sqrt[n]{x^n-1}}{x}\in(0\,;\,1), \quad t^n=1-\frac{1}{x^n} \implies t^nx^n=x^n-1 \implies x^n=\frac{1}{1-t^n} \implies x=\frac{1}{\sqrt[n]{1-t^n}} \\ \mathrm{d}x=-\frac{1}{n}(1-t^n)^{-\frac{1}{n}-1}(-nt^{n-1})\,\mathrm{d}t=\frac{t^{n-1}\,\mathrm{d}t}{\sqrt[n]{(1-t^n)^{n+1}}}, \quad x^n-1=\frac{1}{1-t^n}-1=\frac{t^n}{1-t^n} \implies \sqrt[n]{x^n-1}=t\,(1-t^n)^{-\frac{1}{n}}=\frac{t}{\sqrt[n]{1-t^n}} \end{array}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = -\int \frac{\mathrm{d}t}{t^2-1} = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c = \frac{1}{2}\ln\frac{t+1}{t-1}+c = \frac{1}{2}\ln\left(\sqrt{x^2+1}+x\right)^2+c = \ln\left(\sqrt{x^2+1}+x\right)+c, \qquad \text{for } x>0.$$

$$\boxed{\begin{array}{l} x>0, \quad t=\sqrt{1+\frac{1}{x^2}}=\frac{\sqrt{x^2+1}}{x}>1 \implies x=\frac{1}{\sqrt{t^2-1}}, \quad \mathrm{d}x=-\frac{t\,\mathrm{d}t}{\sqrt{(t^2-1)^3}}, \quad \frac{1}{x^2+1}=\frac{\sqrt{t^2-1}}{t}, \quad \frac{\mathrm{d}x}{\sqrt{x^2+1}}=-\frac{\mathrm{d}t}{t^2-1} \\ t\pm1=\frac{\sqrt{x^2+1}}{x}\pm1=\frac{\sqrt{x^2+1}\pm x}{x} \implies \frac{t+1}{t-1}=\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x}=\frac{\left(\sqrt{x^2+1}+x\right)^2}{(x^2+1)-x^2}=\left(\sqrt{x^2+1}+x\right)^2 \end{array}}$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \boxed{\begin{array}{c} x<0 \implies t=-x>0 \\ \mathrm{d}x=-\mathrm{d}t, \ \sqrt{x^2+1}-x>0 \end{array}} = -\int \frac{\mathrm{d}t}{\sqrt{1+t^2}} = -\ln\left(\sqrt{t^2+1}+t\right)+c = -\ln\left(\sqrt{x^2+1}-x\right)+c =$$

$$= -\ln\left[\left(\sqrt{x^2{+}1}-x\right)\frac{\sqrt{x^2{+}1}+x}{\sqrt{x^2{+}1}+x}\right]+c = -\ln\frac{(x^2+1)-x^2}{\sqrt{x^2{+}1}+x}+c = -\ln\frac{1}{\sqrt{x^2{+}1}+x}+c = \ln\left(\sqrt{x^2{+}1}+x\right)+c, \text{ for } x<0.$$

$$\implies \int \frac{\mathrm{d}x}{\sqrt{x^2+1}} = \ln\left(\sqrt{x^2+1}+x\right)+c, \qquad \text{for } x\in R-\{0\}.$$

$$\int \frac{\mathrm{d}x}{\sqrt[3]{x^3+1}} = \int \frac{\sqrt[3]{t^3-1}}{t} \frac{-t^2\,\mathrm{d}t}{\sqrt[3]{(t^3-1)^4}} = -\int \frac{t\,\mathrm{d}t}{t^3-1} = \boxed{\frac{t}{t^3-1} = \frac{t}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \;\Rightarrow\; A=\frac{1}{3},\ B=-\frac{1}{3},\ C=\frac{1}{3}} =$$

$$\boxed{\begin{gathered} x>0,\quad t=\sqrt[3]{1+\frac{1}{x^3}} = \frac{\sqrt[3]{x^3+1}}{x} > 1,\quad t^3 = 1+\frac{1}{x^3} \implies t^3x^3 = x^3+1 \implies x^3 = \frac{1}{t^3-1} \implies x = \frac{1}{\sqrt[3]{t^3-1}} \\ \mathrm{d}x = -\frac{1}{3}\left(t^3-1\right)^{-\frac{4}{3}} 3t^2\,\mathrm{d}t = \frac{-t^2\,\mathrm{d}t}{\sqrt[3]{(t^3-1)^4}},\quad x^3+1 = \frac{1}{t^3-1}+1 = \frac{t^3}{t^3-1} \implies \sqrt[3]{x^3+1} = t\left(t^3-1\right)^{-\frac{1}{3}} = \frac{t}{\sqrt[3]{t^3-1}} \\ \frac{1}{\sqrt[3]{x^3+1}} = \frac{\sqrt[3]{t^3-1}}{t},\quad t-1 = \frac{\sqrt[3]{x^3+1}}{x} - 1 = \frac{\sqrt[3]{x^3+1}-x}{x},\quad t^3-1 = \frac{1}{x^3},\quad 2t+1 = \frac{2\sqrt[3]{x^3+1}}{x}+1 = \frac{2\sqrt[3]{x^3+1}+x}{x} \end{gathered}}$$

$$= -\int \left[\frac{\frac{1}{3}}{t-1} + \frac{-\frac{t}{3}+\frac{1}{3}}{t^2+t+1}\right]\mathrm{d}x = -\frac{1}{3}\int \frac{\mathrm{d}t}{t-1} + \frac{1}{6}\int \frac{2t-2}{t^2+t+1}\,\mathrm{d}t = -\frac{1}{3}\int \frac{\mathrm{d}t}{t-1} + \frac{1}{6}\int \frac{2t+1}{t^2+t+1}\,\mathrm{d}t - \frac{1}{6}\int \frac{3\,\mathrm{d}t}{t^2+t+1} =$$

$$= -\frac{1}{3}\ln|t-1| + \frac{1}{6}\ln\left|t^2+t+1\right| - \frac{1}{2}\int \frac{\mathrm{d}t}{t^2+t+1} \overset{\text{page}}{\underset{15}{=}} -\frac{1}{3}\ln(t-1) + \frac{1}{6}\ln\left(t^2+t+1\right) - \frac{1}{2}\frac{1}{\sqrt{1-\frac{1}{4}}}\operatorname{arctg}\frac{t+\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} + c =$$

$$= -\frac{1}{3}\ln(t-1) - \frac{1}{6}\ln(t-1) + \frac{1}{6}\ln(t-1) + \frac{1}{6}\ln\left(t^2+t+1\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\left(t+\frac{1}{2}\right)}{\sqrt{3}} + c =$$

$$= -\frac{3}{6}\ln(t-1) + \frac{1}{6}\ln\left[(t-1)(t^2+t+1)\right] - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + c = -\frac{1}{2}\ln(t-1) + \frac{1}{6}\ln\left(t^3-1\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + c =$$

$$= -\frac{1}{2}\ln\frac{\sqrt[3]{x^3+1}-x}{x} + \frac{1}{6}\ln\frac{1}{x^3} - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}} + c = -\frac{1}{2}\ln\frac{\sqrt[3]{x^3+1}-x}{x} + \frac{1}{6}\ln x^{-3} - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}} + c =$$

$$= -\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right) + \frac{1}{2}\ln x + \frac{-3}{6}\ln x - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}} + c =$$

$$= -\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}} + c, \qquad \text{for } x\in(0\,;\,\infty).$$

$$\int \frac{\mathrm{d}x}{\sqrt[3]{x^3+1}} = \boxed{\begin{gathered} -1<x<0,\quad u=-x\in(0;1) \\ \mathrm{d}x = -\mathrm{d}u,\quad x^3+1 = 1-u^3 \end{gathered}} = -\int \frac{\mathrm{d}u}{\sqrt[3]{1-u^3}} \overset{\text{page}}{\underset{70}{=}} -\frac{1}{2}\ln\left(\sqrt[3]{1-u^3}+u\right) + \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{1-u^3}-u}{u\sqrt{3}} + c =$$

$$= -\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right) + \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{-x\sqrt{3}} + c = -\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}} + c,$$

for $x\in(-1\,;\,0)$.

$$\implies \int \frac{\mathrm{d}x}{\sqrt[3]{x^3+1}} = -\frac{1}{2}\ln\left(\sqrt[3]{x^3+1}-x\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{x^3+1}+x}{x\sqrt{3}} + c, \qquad \text{for } x\in(-1\,;\,\infty)-\{0\}.$$

$$\int \frac{\mathrm{d}x}{\sqrt[4]{x^4+1}} = -\int \frac{t^2\,\mathrm{d}t}{t^4-1} = \boxed{\frac{t^2}{t^4-1} = \frac{t^2}{(t-1)(t+1)(t^2+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1} \implies A=\frac{1}{4},\ B=-\frac{1}{4},\ C=0,\ D=\frac{1}{2}} =$$

$$\boxed{\begin{gathered} x>0,\quad t=\sqrt[4]{1+\frac{1}{x^4}} = \frac{\sqrt[4]{x^4+1}}{x} > 1,\quad t^4 = 1+\frac{1}{x^4} \implies t^4x^4 = x^4+1 \implies x^4 = \frac{1}{t^4-1} \implies x = \frac{1}{\sqrt[4]{t^4-1}} \\ \mathrm{d}x = -\frac{1}{4}\left(t^4-1\right)^{-\frac{5}{4}} 4t^3\,\mathrm{d}t = \frac{-t^3\,\mathrm{d}t}{\sqrt[4]{(t^4-1)^5}},\quad x^4+1 = \frac{1}{t^4-1}+1 = \frac{t^4}{t^4-1} \implies \sqrt[4]{x^4+1} = t\left(t^4-1\right)^{-\frac{1}{4}} = \frac{t}{\sqrt[4]{t^4-1}} \\ \frac{1}{\sqrt[4]{x^4+1}} = \frac{\sqrt[4]{t^4-1}}{t},\quad |t\pm1| = t\pm1 = \frac{\sqrt[4]{x^4+1}}{x} \pm 1 = \frac{\sqrt[4]{x^4+1}\pm x}{x},\quad \frac{t+1}{t-1} = \frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x} \end{gathered}}$$

$$= -\int \left[\frac{\frac{1}{4}}{t-1} + \frac{-\frac{1}{4}}{t+1} + \frac{\frac{1}{2}}{t^2+1}\right]\mathrm{d}t = -\frac{1}{4}\ln|t-1| + \frac{1}{4}\ln|t+1| - \frac{1}{2}\operatorname{arctg} t + c = \frac{1}{4}\ln\frac{t+1}{t-1} - \frac{1}{2}\operatorname{arctg} t + c =$$

$$= \frac{1}{4}\ln\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{x} + c, \qquad \text{for } x\in(0\,;\infty).$$

$$\int \frac{\mathrm{d}x}{\sqrt[4]{x^4+1}} = \boxed{x<0,\quad x=-u>0,\quad \mathrm{d}x = -\,\mathrm{d}u} = -\int \frac{\mathrm{d}u}{\sqrt[4]{u^4+1}} = -\frac{1}{4}\ln\frac{\sqrt[4]{u^4+1}+u}{\sqrt[4]{u^4+1}-u} + \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{u^4+1}}{u} + c =$$

$$= -\frac{1}{4}\ln\frac{\sqrt[4]{x^4+1}-x}{\sqrt[4]{x^4+1}+x} + \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{-x} + c = \frac{1}{4}\ln\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{x} + c, \qquad \text{for } x\in(-\infty\,;0).$$

$$\implies \int \frac{\mathrm{d}x}{\sqrt[4]{x^4+1}} = \frac{1}{4}\ln\frac{\sqrt[4]{x^4+1}+x}{\sqrt[4]{x^4+1}-x} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4+1}}{x} + c, \qquad \text{for } x\in R-\{0\}.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \int \frac{\sqrt{1-t^2}}{t}\,\frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)^3}} = -\int \frac{\mathrm{d}t}{t^2-1} = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c = \frac{1}{2}\ln\left|\frac{t+1}{t-1}\right| + c =$$

$$\boxed{\begin{gathered} x>1,\quad t=\sqrt{1-\frac{1}{x^2}} = \frac{\sqrt{x^2-1}}{x} \in (0\,;1) \implies x = \frac{1}{\sqrt{1-t^2}},\ \mathrm{d}x = \frac{t\,\mathrm{d}t}{\sqrt{(1-t^2)^3}},\ x^2-1 = \frac{t^2}{1-t^2},\ \frac{1}{\sqrt{x^2-1}} = \frac{\sqrt{1-t^2}}{t} \\ t\pm1 = \frac{\sqrt{x^2-1}}{x} \pm 1 = \frac{\sqrt{x^2-1}\pm x}{x} \implies \frac{t+1}{t-1} = \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}-x} = \frac{\left(\sqrt{x^2-1}+x\right)^2}{(x^2-1)-x^2} = -\left(\sqrt{x^2-1}+x\right)^2 \end{gathered}}$$

$$= \frac{1}{2}\ln\left|\left(\sqrt{x^2-1}+x\right)^2\right| + c = \frac{2}{2}\ln\left|\sqrt{x^2-1}+x\right| + c = \ln\left|\sqrt{x^2-1}+x\right| + c, \qquad \text{for } x>1.$$

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \boxed{x<-1,\ u=-x>1,\ \mathrm{d}x = -\,\mathrm{d}t} = -\int \frac{\mathrm{d}u}{\sqrt{u^2-1}} = -\ln\left|\sqrt{u^2-1}+u\right| + c = -\ln\left|\sqrt{x^2-1}-x\right| + c =$$

$$= -\ln\left|\left(\sqrt{x^2-1}-x\right)\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}+x}\right| + c = -\ln\left|\frac{(x^2-1)-x^2}{\sqrt{x^2-1}+x}\right| + c = -\ln\left|\frac{1}{\sqrt{x^2-1}+x}\right| + c = \ln\left|\sqrt{x^2-1}+x\right| + c,$$

for $x<-1$.

$$\implies \int \frac{\mathrm{d}x}{\sqrt{x^2-1}} = \ln\left|\sqrt{x^2-1}+x\right| + c, \qquad \text{for } x\in R-\langle -1\,;1\rangle = (-\infty\,;-1)\cup(1\,;\infty).$$

$$\int \frac{dx}{\sqrt[3]{x^3-1}} = \int \frac{t\,dt}{1-t^3} = \int \left[\frac{-\frac{1}{3}}{t-1} + \frac{\frac{t}{3}-\frac{1}{3}}{t^2+t+1} \right] dt = -\frac{1}{3}\int \frac{dt}{t-1} + \frac{1}{6}\int \frac{2t-2}{t^2+t+1}\,dt = -\frac{1}{3}\ln|t-1| + \frac{1}{6}\int \frac{2t+1-3}{t^2+t+1}\,dt =$$

$$\boxed{\begin{array}{c} x>1, \quad t=\sqrt[3]{1-\frac{1}{x^3}} = \frac{\sqrt[3]{x^3-1}}{x} \in (0\,;1), \quad t^3 = 1-\frac{1}{x^3} \implies 1-t^3=\frac{1}{x^3},\ x^3=\frac{1}{1-t^3} \implies x=\frac{1}{\sqrt[3]{1-t^3}},\ dx=\frac{t^2\,dt}{\sqrt[3]{(1-t^3)^4}} \\ x^3-1=\frac{t^3}{1-t^3} \implies \sqrt[3]{x^3-1}=\frac{t}{\sqrt[3]{1-t^3}},\ 1-t=1-\frac{\sqrt[3]{x^3-1}}{x}=\frac{x-\sqrt[3]{x^3-1}}{x},\ 2t+1=\frac{2\sqrt[3]{x^3-1}}{x}+1=\frac{2\sqrt[3]{x^3-1}+x}{x} \\ \hline \frac{t}{1-t^3}=\frac{-t}{(t-1)(t^2+t+1)}=\frac{A}{t-1}+\frac{Bt+C}{t^2+t+1} \implies A=-\frac{1}{3},\ B=\frac{1}{3},\ C=-\frac{1}{3} \end{array}}$$

$$= -\frac{1}{3}\ln|t-1| + \frac{1}{6}\int \frac{2t+1}{t^2+t+1}\,dt - \frac{1}{2}\int \frac{dt}{t^2+t+1} \underset{15}{\overset{\text{page}}{=}} -\frac{1}{3}\ln|t-1| + \frac{1}{6}\ln\left|t^2+t+1\right| - \frac{1}{2}\,\frac{1}{\sqrt{1-\frac{1}{4}}}\operatorname{arctg}\frac{t+\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} + c =$$

$$= -\frac{1}{3}\ln(1-t) - \frac{1}{6}\ln(1-t) + \frac{1}{6}\ln(1-t) + \frac{1}{6}\ln\left(t^2+t+1\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2(t+\frac{1}{2})}{\sqrt{3}} + c =$$

$$= -\frac{3}{6}\ln(1-t) + \frac{1}{6}\ln\left[(1-t)(t^2+t+1)\right] - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + c = -\frac{1}{2}\ln(1-t) + \frac{1}{6}\ln\left(1-t^3\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t+1}{\sqrt{3}} + c =$$

$$= -\frac{1}{2}\ln\frac{x-\sqrt[3]{x^3-1}}{x} + \frac{1}{6}\ln\frac{1}{x^3} - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{x^3-1}+x}{x\sqrt{3}} + c = -\frac{1}{2}\ln\left(x-\sqrt[3]{x^3-1}\right) + \frac{1}{2}\ln x - \frac{1}{6}\ln x^3 - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{x^3-1}+x}{x\sqrt{3}} + c =$$

$$= -\frac{1}{2}\ln\left(x-\sqrt[3]{x^3-1}\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{x^3-1}+x}{x\sqrt{3}} + c, \qquad \text{for } x\in(1\,;\infty).$$

$$\int \frac{dx}{\sqrt[4]{x^4-1}} = \int \frac{t^2\,dt}{1-t^4} = \int \left[\frac{-\frac{1}{4}}{t-1} + \frac{\frac{1}{4}}{t+1} + \frac{-\frac{1}{2}}{t^2+1} \right] dt = -\frac{1}{4}\ln|t-1| + \frac{1}{4}\ln|t+1| - \frac{1}{2}\operatorname{arctg} t + c =$$

$$\boxed{\begin{array}{c} x>1, \quad t=\sqrt[4]{1-\frac{1}{x^4}}=\frac{\sqrt[4]{x^4-1}}{x}\in(0\,;1),\ t^4=1-\frac{1}{x^4} \implies x^4=\frac{1}{1-t^4},\ x=\frac{1}{\sqrt[4]{1-t^4}},\ dx=\frac{t^3\,dt}{\sqrt[4]{(1-t^4)^5}}, \\ x^4-1=\frac{t^4}{1-t^4},\ \sqrt[4]{x^4-1}=\frac{t}{\sqrt[4]{1-t^4}},\ |1\pm t|=1\pm t=1\pm\frac{\sqrt[4]{x^4-1}}{x}=\frac{x\pm\sqrt[4]{x^4-1}}{x},\ \frac{1+t}{1-t}=\frac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} \\ \hline \frac{t^2}{1-t^4}=\frac{-t^2}{(t-1)(t+1)(t^2+1)}=\frac{A}{t-1}+\frac{B}{t+1}+\frac{Ct+D}{t^2+1} \implies A=-\frac{1}{4},\ B=\frac{1}{4},\ C=0,\ D=-\frac{1}{2} \end{array}}$$

$$= \frac{1}{4}\ln\left|\frac{t+1}{t-1}\right| - \frac{1}{2}\operatorname{arctg} t + c = \frac{1}{4}\ln\frac{1+t}{1-t} - \frac{1}{2}\operatorname{arctg} t + c = \frac{1}{4}\ln\frac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4-1}}{x} + c, \qquad \text{for } x\in(1\,;\infty).$$

$$\int \frac{dx}{\sqrt[4]{x^4-1}} = \boxed{x<-1,\ u=-x>1,\ dx=-du} = -\int \frac{du}{\sqrt[4]{u^4-1}} = -\frac{1}{4}\ln\frac{u+\sqrt[4]{u^4-1}}{u-\sqrt[4]{u^4-1}} + \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{u^4-1}}{u} + c =$$

$$= -\frac{1}{4}\ln\frac{-x+\sqrt[4]{x^4-1}}{-x-\sqrt[4]{x^4-1}} + \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4-1}}{-x} + c = -\frac{1}{4}\ln\frac{x-\sqrt[4]{x^4-1}}{x+\sqrt[4]{x^4-1}} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4-1}}{x} + c =$$

$$= \frac{1}{4}\ln\frac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4-1}}{x} + c, \qquad \text{for } x\in(-\infty\,;-1).$$

$$\implies \int \frac{dx}{\sqrt[4]{x^4-1}} = \frac{1}{4}\ln\frac{x+\sqrt[4]{x^4-1}}{x-\sqrt[4]{x^4-1}} - \frac{1}{2}\operatorname{arctg}\frac{\sqrt[4]{x^4-1}}{x} + c, \qquad \text{for } x\in(-\infty\,;-1)\cup(1\,;\infty).$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\sqrt{t^2+1}}{t}\,\frac{-t\,dt}{\sqrt{(t^2+1)^3}} = -\int \frac{dt}{t^2+1} = -\operatorname{arctg} t + c_1 = -\operatorname{arctg}\frac{\sqrt{1-x^2}}{x} + c_1, \qquad \text{for } x\in(0\,;\,1).$$

$$\boxed{\begin{array}{l} 0<x<1,\quad t=\sqrt{\frac{1}{x^2}-1}=\frac{\sqrt{1-x^2}}{x}>0,\quad t^2=\frac{1}{x^2}-1 \implies t^2x^2=1-x^2 \implies x^2=\frac{1}{t^2+1} \implies x=\frac{1}{\sqrt{t^2+1}} \\ dx=-\frac{1}{2}\left(t^2+1\right)^{-\frac{3}{2}}2t\,dt=\frac{-t\,dt}{\sqrt{(t^2+1)^3}},\quad 1-x^2=1-\frac{1}{t^2+1}=\frac{t^2}{t^2+1},\quad \sqrt{1-x^2}=\frac{t}{\sqrt{t^2+1}},\quad \frac{1}{x^2+1}=\frac{\sqrt{t^2+1}}{t} \end{array}}$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \boxed{\begin{array}{c} -1<x<0,\ \ u=-x>0 \\ dx=-du \end{array}} = -\int \frac{du}{\sqrt{1-u^2}} = \operatorname{arctg}\frac{\sqrt{1-u^2}}{u} + c_1 = \operatorname{arctg}\frac{\sqrt{1-x^2}}{-x} + c_1 = -\operatorname{arctg}\frac{\sqrt{1-x^2}}{x} + c_1,$$

for $x\in(-1\,;\,0)$.

$$\Longrightarrow \int \frac{dx}{\sqrt{1-x^2}} = -\operatorname{arctg}\frac{\sqrt{1-x^2}}{x} + c_1, \qquad \text{for } x\in(-1\,;\,1)-\{0\},$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \boxed{\begin{array}{l} \text{2-nd Euler: } -1<x<1,\ \sqrt{1-x^2}=xt+1 \ \Rightarrow\ 1-x^2=x^2t^2+2tx+1 \ \Rightarrow\ -2tx=x^2+x^2t^2 \ \xrightarrow{x\neq 0}\ x=\frac{-2t}{t^2+1} \\ t=\frac{\sqrt{1-x^2}-1}{x},\quad \sqrt{1-x^2}=\frac{-2t^2}{t^2+1}+1=\frac{1-t^2}{t^2+1},\quad dx=\frac{-2(t^2+1)+2t\cdot 2t}{(t^2+1)^2}\,dt=\frac{2(t^2-1)}{(t^2+1)^2}\,dt \end{array}} =$$

$$= \int \frac{t^2+1}{1-t^2}\,\frac{2(t^2-1)}{(t^2+1)^2}\,dt = -2\int \frac{dt}{t^2+1} = -2\operatorname{arctg} t + c_2 = -2\operatorname{arctg}\frac{\sqrt{1-x^2}-1}{x} + c_2, \qquad \text{for } x\in(-1\,;\,1)-\{0\}.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \boxed{\begin{array}{c} -1<x<1,\ x=\sin t,\ t=\arcsin x\in(-\pi/2\,;\,\pi/2) \\ dx=\cos t\,dt,\ \sqrt{1-x^2}=\sqrt{1-\sin^2 t}=\sqrt{\cos^2 t}=|\cos t|=\cos t \end{array}} = \int \frac{\cos t\,dt}{\cos t} = \int dt = t+c_3 = \arcsin x + c_3,$$

for $x\in(-1\,;\,1)$.

$$\int \frac{dx}{\sqrt{1-x^2}} = \boxed{\begin{array}{c} -1<x<1,\ x=\cos t,\ t=\arccos x\in(0\,;\,\pi),\ dx=-\sin t\,dt \\ \sqrt{1-x^2}=\sqrt{1-\cos^2 t}=\sqrt{\sin^2 t}=|\sin t|=\sin t,\ t\in(0\,;\,\pi) \end{array}} = -\int \frac{\sin t\,dt}{\sin t} = -\int dt = -t+c_4 = -\arccos x + c_4,$$

for $x\in(-1\,;\,1)$.

$$\Longrightarrow \int \frac{dx}{\sqrt{1-x^2}} = -\operatorname{arctg}\frac{\sqrt{1-x^2}}{x} + c_1 = -2\operatorname{arctg}\frac{\sqrt{1-x^2}-1}{x} + c_2, \qquad \text{for } x\in(-1\,;\,1)-\{0\},$$

$$= \arcsin x + c_3 = -\arccos x + c_4, \qquad \text{for } x\in(-1\,;\,1).$$

$$\int \frac{dx}{\sqrt[3]{1-x^3}} = -\int \frac{t\,dt}{t^3+1} = \int \left[\frac{\frac{1}{3}}{t+1} + \frac{-\frac{t}{3}-\frac{1}{3}}{t^2-t+1}\right] dt = \frac{1}{3}\int \frac{dt}{t+1} - \frac{1}{6}\int \frac{2t+2}{t^2-t+1}\,dt = \frac{1}{3}\ln|t+1| - \frac{1}{6}\int \frac{2t-1+3}{t^2-t+1}\,dt =$$

$$\boxed{\begin{array}{c} 0<x<1,\ t=\sqrt[3]{\frac{1}{x^3}-1}=\frac{\sqrt[3]{1-x^3}}{x}>0,\ t^3=\frac{1}{x^3}-1 \implies t^3+1=\frac{1}{x^3},\ x^3=\frac{1}{t^3+1},\ x=\frac{1}{\sqrt[3]{t^3+1}},\ dx=\frac{-t^2\,dt}{\sqrt[3]{(t^3+1)^4}} \\ 1-x^3=\frac{t^3}{t^3+1},\ \sqrt[3]{1-x^3}=\frac{t}{\sqrt[3]{t^3+1}},\ t+1=\frac{\sqrt[3]{1-x^3}}{x}+1=\frac{\sqrt[3]{1-x^3}+x}{x},\ 2t-1=\frac{2\sqrt[3]{1-x^3}}{x}-1=\frac{2\sqrt[3]{1-x^3}-x}{x} \\ \hline -\frac{t}{t^3+1}=\frac{-t}{(t+1)(t^2-t+1)}=\frac{A}{t+1}+\frac{Bt+C}{t^2-t+1} \implies A=\frac{1}{3},\ B=-\frac{1}{3},\ C=-\frac{1}{3} \end{array}}$$

$$= \frac{1}{3}\ln(t+1) - \frac{1}{6}\int \frac{2t-1}{t^2-t+1}\,dt - \frac{1}{2}\int \frac{dt}{t^2-t+1} \overset{\text{page}}{\underset{15}{=}} \frac{1}{3}\ln(t+1) - \frac{1}{6}\ln\left|t^2-t+1\right| - \frac{1}{2}\frac{1}{\sqrt{1-\frac{1}{4}}}\operatorname{arctg}\frac{t-\frac{1}{2}}{\sqrt{1-\frac{1}{4}}} + c =$$

$$= \frac{1}{3}\ln(t+1) - \frac{1}{6}\ln\left(t^2-t+1\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2(t-\frac{1}{2})}{\sqrt{3}} + c = \frac{1}{3}\ln(t+1) - \frac{1}{6}\ln\frac{t^3+1}{t+1} - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + c =$$

$$= \frac{1}{3}\ln(t+1) - \frac{1}{6}\ln\left(t^3+1\right) + \frac{1}{6}\ln(t+1) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + c = \frac{3}{6}\ln(t+1) - \frac{1}{6}\ln\left(t^3+1\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2t-1}{\sqrt{3}} + c =$$

$$= \frac{1}{2}\ln\frac{\sqrt[3]{1-x^3}+x}{x} - \frac{1}{6}\ln\frac{1}{x^3} - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}} + c = \frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right) - \frac{1}{2}\ln x + \frac{1}{6}\ln x^3 - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}} + c =$$

$$= \frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right) - \frac{\sqrt{3}}{3}\operatorname{arctg}\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}} + c, \qquad \text{for } x\in(0\,;\,1).$$

$$\int \frac{dx}{\sqrt[3]{1-x^3}} = \boxed{\begin{array}{c} x<0,\ u=-x>0,\ dx=-du \\ 1-x^3=1-(-u)^3=u^3+1 \end{array}} = -\int \frac{du}{\sqrt[3]{u^3+1}} \overset{\text{page}}{\underset{66}{=}} \frac{1}{2}\ln\left(\sqrt[3]{u^3+1}-u\right) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{u^3+1}+u}{u\sqrt{3}} + c =$$

$$= \frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right) + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{1-x^3}-x}{-x\sqrt{3}} + c = \frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}} + c, \quad \text{for } x\in(-\infty\,;\,0).$$

$$\implies \int \frac{dx}{\sqrt[3]{1-x^3}} = \frac{1}{2}\ln\left(\sqrt[3]{1-x^3}+x\right) - \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt[3]{1-x^3}-x}{x\sqrt{3}} + c, \qquad \text{for } x\in(-\infty\,;\,1)-\{0\}.$$

$$\int \frac{\mathrm{d}x}{\sqrt[4]{1-x^4}} = \int \frac{-t^2\,\mathrm{d}t}{t^4+1} = \int \left[\frac{\frac{\sqrt{2}}{4}t}{t^2+\sqrt{2}t+1} + \frac{-\frac{\sqrt{2}}{4}t}{t^2-\sqrt{2}t+1}\right]\mathrm{d}t = \frac{\sqrt{2}}{8}\int \left[\frac{2t}{t^2+\sqrt{2}t+1} - \frac{2t}{t^2-\sqrt{2}t+1}\right]\mathrm{d}t =$$

$$\boxed{\begin{array}{l} 0<x<1,\ t=\sqrt[4]{\frac{1}{x^4}-1}=\frac{\sqrt[4]{1-x^4}}{x}>0,\ t^4=\frac{1}{x^4}-1 \implies x^4=\frac{1}{t^4+1},\ x=\frac{1}{\sqrt[4]{t^4+1}},\ \mathrm{d}x=\frac{-t^3\,\mathrm{d}t}{\sqrt[4]{(t^4+1)^5}},\ \sqrt[4]{1-x^4}=\frac{t}{\sqrt[4]{t^4+1}} \\ \left|t^2\pm\sqrt{2}t+1\right| = t^2\pm\sqrt{2}t+1 = \frac{\sqrt{1-x^4}}{x^2}\pm\sqrt{2}\frac{\sqrt[4]{1-x^4}}{x}+1 = \frac{\sqrt{1-x^4}\pm\sqrt{2}x\sqrt[4]{1-x^4}+x^2}{x^2} = \frac{\sqrt{1-x^4}\pm x\sqrt[4]{4(1-x^4)}+x^2}{x^2} \\ \frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} = \frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2},\qquad \sqrt{2}t\pm1=\sqrt{2}\frac{\sqrt[4]{1-x^4}}{x}\pm1=\frac{\sqrt{2}\sqrt[4]{1-x^4}\pm x}{x}=\frac{\sqrt[4]{4(1-x^4)}\pm x}{x} \\ \hline \frac{-t^2}{t^4+1}=\frac{-t^2}{(t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)}=\frac{At+B}{t^2+\sqrt{2}t+1}+\frac{Ct+D}{t^2-\sqrt{2}t+1} \implies A=\frac{\sqrt{2}}{4},\ B=0,\ C=-\frac{\sqrt{2}}{4},\ D=0 \end{array}}$$

$$= \frac{\sqrt{2}}{8}\int\left[\frac{2t+\sqrt{2}-\sqrt{2}}{t^2+\sqrt{2}t+1} - \frac{2t-\sqrt{2}+\sqrt{2}}{t^2-\sqrt{2}t+1}\right]\mathrm{d}t = \frac{\sqrt{2}}{8}\int\left[\frac{2t+\sqrt{2}}{t^2+\sqrt{2}t+1} - \frac{2t-\sqrt{2}}{t^2-\sqrt{2}t+1}\right]\mathrm{d}t + \frac{2}{8}\int\left[\frac{-1}{t^2+\sqrt{2}t+1} - \frac{1}{t^2-\sqrt{2}t+1}\right]\mathrm{d}t =$$

$$= \frac{\sqrt{2}}{8}\left[\ln\left|t^2+\sqrt{2}t+1\right| - \ln\left|t^2-\sqrt{2}t+1\right|\right] - \frac{1}{4}\int\frac{\mathrm{d}t}{t^2+\sqrt{2}t+1} - \frac{1}{4}\int\frac{\mathrm{d}t}{t^2-\sqrt{2}t+1} \overset{\text{page}}{\underset{15}{=}}$$

$$= \frac{\sqrt{2}}{8}\ln\left|\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1}\right| - \frac{1}{4}\frac{1}{\sqrt{1-\frac{2}{4}}}\operatorname{arctg}\frac{t+\frac{\sqrt{2}}{2}}{\sqrt{1-\frac{2}{4}}} - \frac{1}{4}\frac{1}{\sqrt{1-\frac{2}{4}}}\operatorname{arctg}\frac{t-\frac{\sqrt{2}}{2}}{\sqrt{1-\frac{2}{4}}} + c =$$

$$= \frac{\sqrt{2}}{8}\ln\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} - \frac{1}{2\sqrt{2}}\left[\operatorname{arctg}\frac{t+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} + \operatorname{arctg}\frac{t-\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right] + c = \frac{\sqrt{2}}{8}\ln\frac{t^2+\sqrt{2}t+1}{t^2-\sqrt{2}t+1} - \frac{\sqrt{2}}{4}\left[\operatorname{arctg}(\sqrt{2}t+1) + \operatorname{arctg}(\sqrt{2}t-1)\right] + c =$$

$$= \frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{x} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{x} + c, \qquad \text{for } x\in(0\,;\,1).$$

$$\int\frac{\mathrm{d}x}{\sqrt[4]{1-x^4}} = \boxed{x<-1,\ u=-x>1,\ \mathrm{d}x=-\mathrm{d}u} = -\int\frac{\mathrm{d}u}{\sqrt[4]{1-u^4}} =$$

$$= -\frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-u^4}+u\sqrt[4]{4(1-u^4)}+u^2}{\sqrt{1-u^4}-u\sqrt[4]{4(1-u^4)}+u^2} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-u^4)}+u}{u} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-u^4)}-u}{u} + c =$$

$$= -\frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{-x} + \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{-x} + c =$$

$$= \frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{4(1-x^4)}+x^2} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{x} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{x} + c, \qquad \text{for } x\in(-1\,;\,0).$$

$$\implies \int\frac{\mathrm{d}x}{\sqrt[4]{1-x^4}} = \frac{\sqrt{2}}{8}\ln\frac{\sqrt{1-x^4}+x\sqrt[4]{4(1-x^4)}+x^2}{\sqrt{1-x^4}-x\sqrt[4]{(1-x^4)}+x^2} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}+x}{x} - \frac{\sqrt{2}}{4}\operatorname{arctg}\frac{\sqrt[4]{4(1-x^4)}-x}{x} + c, \text{ for } x\in(-1\,;\,1).$$

$$\int \sin nx\,\mathrm{d}x = \boxed{nx=t,\ \ \mathrm{d}x=\frac{\mathrm{d}t}{n}} = \frac{1}{n}\int \sin t\,\mathrm{d}t = -\frac{\cos t}{n}+c = -\frac{\cos nx}{n}+c, \qquad \text{for } n\in N,\ x\in R.$$

$$\int x\sin nx\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x & \Rightarrow\ u'=1 \\ v'=\sin nx & \Rightarrow\ v=-\dfrac{\cos nx}{n}\end{array}} = -\frac{x\cos nx}{n}+\int\frac{\cos nx}{n}\,\mathrm{d}x = -\frac{x\cos nx}{n}+\frac{1}{n}\frac{\sin nx}{n}+c = \frac{\sin nx}{n^2}-\frac{x\cos nx}{n}+c,$$

for $x\in R,\ n\in N$.

$$\int x\sin nx\,\mathrm{d}x = [A+Bx]\sin nx+[C+Dx]\cos nx+c = \left[\frac{1}{n^2}+0\right]\sin nx+\left[0-\frac{x}{n}\right]\cos nx+c = \frac{\sin nx}{n^2}-\frac{x\cos nx}{n}+c, \ \text{ for } x\in R,\ n\in N.$$

$$\xRightarrow{\text{Derivation}}\quad \left[\int x\sin nx\,\mathrm{d}x\right]' = [A+Bx]'\sin nx+[A+Bx]\sin' nx+[C+Dx]'\cos nx+[C+Dx]\cos' nx$$

$$x\sin nx = B\sin nx+[A+Bx]n\cos nx+D\cos nx-[C+Dx]n\sin nx =$$

$$= [B-Cn-Dnx]\sin nx+[An+D+Bnx]\cos nx$$

$$\xRightarrow{\text{Equations}}\quad 0=B-Cn,\ \ x=-Dnx,\ \ 0=D+An,\ \ 0=Bnx \qquad \Longrightarrow\ A=\frac{1}{n^2},\ \ B=0,\ \ C=0,\ \ D=-\frac{1}{n}$$

$$\int x^2\sin nx\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x^2 & \Rightarrow\ u'=2x \\ v'=\sin nx & \Rightarrow\ v=-\dfrac{\cos nx}{n}\end{array}} = -\frac{x^2\cos nx}{n}+\frac{2}{n}\int x\cos nx\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x & \Rightarrow\ u'=1 \\ v'=\cos nx & \Rightarrow\ v=\dfrac{\sin nx}{n}\end{array}} =$$

$$= -\frac{x^2\cos nx}{n}+\frac{2}{n}\left[\frac{x\sin nx}{n}-\frac{1}{n}\int\sin nx\,\mathrm{d}x\right] = -\frac{x^2\cos nx}{n}+\frac{2}{n}\left[\frac{x\sin nx}{n}-\frac{1}{n}\frac{-\cos nx}{n}\right] = \frac{2x\sin nx}{n^2}+\frac{2\cos nx}{n^3}-\frac{x^2\cos nx}{n}+c,$$

for $x\in R,\ n\in N$.

$$\int x^2\sin nx\,\mathrm{d}x = [A+Bx+Cx^2]\sin nx+[D+Ex+Fx^2]\cos nx+c = \frac{2x}{n^2}\sin nx+\left[\frac{2}{n^3}-\frac{x^2}{n}\right]\cos nx+c, \qquad \text{for } x\in R,\ n\in N.$$

$$\xRightarrow{\text{Derivation}}\quad \left[\int x^2\sin nx\,\mathrm{d}x\right]' = [B+2Cx]\sin nx+[A+Bx+Cx^2]n\cos nx+[E+2Fx]\cos nx-[D+Ex+Fx^2]n\sin nx$$

$$x^2\sin nx = [B-Dn+(2C-En)x-Fnx^2]\sin nx+[An+E+(Bn+2F)x+Cnx^2]\cos nx$$

$$\xRightarrow{\text{Equations}}\quad 0=B-Dn,\ \ 0=(2C-En)x,\ \ x^2=-Fnx^2,\ \ 0=E+An,\ \ 0=(2F+Bn)x,\ \ 0=Cnx^2$$

$$\Longrightarrow\ A=0,\ \ B=\frac{2}{n^2},\ \ C=0,\ \ D=\frac{2}{n^3},\ \ E=0,\ \ F=-\frac{1}{n}$$

$$\int x^3\sin nx\,\mathrm{d}x = [A+Bx+Cx^2+Dx^3]\sin nx+[E+Fx+Gx^2+Hx^3]\cos nx+c = \left[-\frac{6}{n^4}+\frac{3x^2}{n^2}\right]\sin nx+\left[\frac{6x}{n^3}-\frac{x^3}{n}\right]\cos nx+c,$$

for $x\in R,\ n\in N$.

$$\xRightarrow{\text{Derivation}}\quad \left[\int x^3\sin nx\,\mathrm{d}x\right]' = [B+2Cx+3Dx^2]\sin nx+[A+Bx+Cx^2+Dx^3]n\cos nx+$$

$$+[F+2Gx+3Hx^2]\cos nx-[E+Fx+Gx^2+Hx^3]n\sin nx$$

$$x^3\sin nx = [B-En+(2C-Fn)x-(3D-Gn)x^2-Hnx^3]\sin nx+[An+F+(Bn+2G)x+(Cn+3H)x^2+Dnx^3]\cos nx$$

$$\xRightarrow{\text{Equations}}\quad 0=B-En,\ 0=(2C-Fn)x,\ 0=(3D-Gn)x^2,\ x^3=-Hnx^3,\ 0=F+An,\ 0=(2G+Bn)x,\ 0=(3H+Cn)x^2,\ 0=Dnx^3$$

$$\Longrightarrow\ A=-\frac{6}{n^4},\ \ B=0,\ \ C=\frac{3}{n^2},\ \ D=0,\ \ E=0,\ \ F=\frac{6}{n^3},\ \ G=0,\ \ H=-\frac{1}{n}$$

$$\int \cos nx\,\mathrm{d}x = \boxed{nx=t,\ \ \mathrm{d}x=\frac{\mathrm{d}t}{n}} = \frac{1}{n}\int \cos t\,\mathrm{d}t = \frac{\sin t}{n}+c = \frac{\sin nx}{n}+c, \qquad \text{for } \ n\in N,\ x\in R.$$

$$\int x\cos nx\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x & \Rightarrow\ u'=1 \\ v'=\cos nx & \Rightarrow\ v=\dfrac{\sin nx}{n}\end{array}} = \frac{x\sin nx}{n} - \int \frac{\sin nx}{n}\,\mathrm{d}x = \frac{x\sin nx}{n} - \frac{1}{n}\,\frac{-\cos nx}{n}+c = \frac{\cos nx}{n^2}+\frac{x\sin nx}{n}+c,$$

for $x\in R,\ n\in N$.

$$\int x\cos nx\,\mathrm{d}x = [A+Bx]\cos nx + [C+Dx]\sin nx + c = \left[\frac{1}{n^2}+0\right]\cos nx + \left[0+\frac{x}{n}\right]\sin nx + c = \frac{\cos nx}{n^2}+\frac{x\sin nx}{n}+c, \quad \text{for } x\in R,\ n\in N.$$

$$\xRightarrow{\text{Derivation}} \quad \left[\int x\cos nx\,\mathrm{d}x\right]' = [A+Bx]'\cos nx + [A+Bx]\cos' nx + [C+Dx]'\sin nx + [C+Dx]\sin' nx$$

$$x\cos nx = B\cos nx - [A+Bx]n\sin nx + D\sin nx + [C+Dx]n\cos nx =$$

$$= [B+Cn+Dnx]\cos nx + [-An+D-Bnx]\sin nx$$

$$\xRightarrow{\text{Equations}} \quad 0=B+Cn,\ \ x=Dnx,\ \ 0=D-An,\ \ 0=-Bnx \qquad \Longrightarrow A=\frac{1}{n^2},\ B=0,\ C=0,\ D=\frac{1}{n}$$

$$\int x^2\cos nx\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x^2 & \Rightarrow\ u'=2x \\ v'=\cos nx & \Rightarrow\ v=\dfrac{\sin nx}{n}\end{array}} = \frac{x^2\sin nx}{n} - \frac{2}{n}\int x\sin nx\,\mathrm{d}x = \boxed{\begin{array}{ll} u=x & \Rightarrow\ u'=1 \\ v'=\sin nx & \Rightarrow\ v=-\dfrac{\cos nx}{n}\end{array}} =$$

$$= \frac{x^2\sin nx}{n} - \frac{2}{n}\left[-\frac{x\cos nx}{n} + \frac{1}{n}\int \cos nx\,\mathrm{d}x\right] = \frac{x^2\sin nx}{n} - \frac{2}{n}\left[-\frac{x\cos nx}{n} + \frac{1}{n}\,\frac{\sin nx}{n}\right] = \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3} + \frac{x^2\sin nx}{n} + c,$$

for $x\in R,\ n\in N$.

$$\int x^2\cos nx\,\mathrm{d}x = [A+Bx+Cx^2]\cos nx + [D+Ex+Fx^2]\sin nx + c = \frac{2x}{n^2}\cos nx + \left[-\frac{2}{n^3}+\frac{x^2}{n}\right]\sin nx + c, \qquad \text{for } \ x\in R,\ n\in N.$$

$$\xRightarrow{\text{Derivation}} \quad \left[\int x^2\cos nx\,\mathrm{d}x\right]' = [B+2Cx]\cos nx - [A+Bx+Cx^2]n\sin nx + [E+2Fx]\sin nx + [D+Ex+Fx^2]n\cos nx$$

$$x^2\cos nx = [B+Dn+(2C+En)x+Fnx^2]\cos nx + [-An+E+(-Bn+2F)x-Cnx^2]\sin nx$$

$$\xRightarrow{\text{Equations}} \quad 0=B+Dn,\ \ 0=(2C+En)x,\ \ x^2=Fnx^2,\ \ 0=E-An,\ \ 0=(2F-Bn)x,\ \ 0=-Cnx^2$$

$$\Longrightarrow A=0,\ \ B=\frac{2}{n^2},\ \ C=0,\ \ D=-\frac{2}{n^3},\ \ E=0,\ \ F=\frac{1}{n}$$

$$\int x^3\cos nx\,\mathrm{d}x = [A+Bx+Cx^2+Dx^3]\cos nx + [E+Fx+Gx^2+Hx^3]\sin nx + c = \left[-\frac{6}{n^4}+\frac{3x^2}{n^2}\right]\cos nx + \left[-\frac{6x}{n^3}+\frac{x^3}{n}\right]\sin nx + c,$$

for $x\in R,\ n\in N$.

$$\xRightarrow{\text{Derivation}} \quad \left[\int x^3\cos nx\,\mathrm{d}x\right]' = [B+2Cx+3Dx^2]\cos nx - [A+Bx+Cx^2+Dx^3]n\sin nx +$$

$$+[F+2Gx+3Hx^2]\sin nx + [E+Fx+Gx^2+Hx^3]n\cos nx$$

$$x^3\cos nx = [B+En+(2C+Fn)x-(3D+Gn)x^2+Hnx^3]\cos nx + [-An+F+(-Bn+2G)x+(-Cn+3H)x^2-Dnx^3]\sin nx$$

$$\xRightarrow{\text{Equations}} \quad 0=B+En,\ 0=(2C+Fn)x,\ 0=(3D+Gn)x^2,\ x^3=Hnx^3,\ 0=F-An,\ 0=(2G-Bn)x,\ 0=(3H-Cn)x^2,\ 0=-Dnx^3$$

$$\Longrightarrow A=-\frac{6}{n^4},\ B=0,\ C=\frac{3}{n^2},\ D=0,\ E=0,\ F=-\frac{6}{n^3},\ G=0,\ H=\frac{1}{n}$$